C 13.22/sec. B:74 B/2

SEP 10 1970
SECTION B
Public Documents Division
Oklahoma Department of Library

A UNITED STATES DEPARTMENT OF COMMERCE PUBLICATION

JOURNAL OF RESEARCH
of the NATIONAL BUREAU OF STANDARDS Volume 74B, Number 2 April-June 1970

Mathematical Sciences

U.S. DEPARTMENT OF COMMERCE
National Bureau of Standards

Journal of Research

of the

National Bureau of Standards

B. Mathematical Sciences

Apr.–June • 1970

VOLUME • 74B

NUMBER • 2

Editor: Morris Newman

Associate Editors: F.W. Olver and J. R. Edmonds

Publication dates: Mar. 10, June 10, Sept. 10, Dec. 10, 1970

J. Res. Nat. Bur. Stand. (U.S.), B CODEN: JNBEA 74B(2) 67-129 (1970)

JOURNAL OF RESEARCH

The National Bureau of Standards Journal of Research reports research and development in the fields of activity shown at right. Also included from time to time are survey articles on topics closely related to the Bureau's scientific and technical program.

The Journal is published in three separate sections as follows:

A. Physics and Chemistry

Contains papers of interest primarily to scientists working in these fields. Issued six times a year. Annual subscription: domestic, $9.50; foreign, $11.75. Single copy, varies.

B. Mathematical Sciences

Presents studies and compilations designed mainly for the mathematician and the theoretical physicist. Issued quarterly. Annual subscription: domestic, $5.00; foreign, $6.25. Single copy $1.25.

C. Engineering and Instrumentation

Reports research and development results of interest chiefly to the engineer and the applied scientist. Issued quarterly. Annual subscription: domestic, $5.00; foreign, $6.25. Single copy $1.25.

The National Bureau of Standards [1] was established by an act of Congress March 3, 1901. Today, in addition to serving as the Nation's central measurement laboratory, the Bureau is a principal focal point in the Federal Government for assuring maximum application of the physical and engineering sciences to the advancement of technology in industry and commerce. To this end the Bureau conducts research and provides central national services in four broad program areas. These are: (1) basic measurements and standards, (2) materials measurements and standards, (3) technological measurements and standards, and (4) transfer of technology.

The Bureau comprises the Institute for Basic Standards, the Institute for Materials Research, the Institute for Applied Technology, the Center for Radiation Research, the Center for Computer Sciences and Technology, and the Office for Information Programs.

THE INSTITUTE FOR BASIC STANDARDS provides the central basis within the United States of a complete and consistent system of physical measurement; coordinates that system with measurement systems of other nations; and furnishes essential services leading to accurate and uniform physical measurements throughout the Nation's scientific community, industry, and commerce. The Institute consists of an Office of Measurement Services and the following technical divisions:

Applied Mathematics—Electricity—Metrology—Mechanics—Heat—Atomic and Molecular Physics—Radio Physics [2]—Radio Engineering [2]—Time and Frequency [2]—Astrophysics [2]—Cryogenics.[2]

THE INSTITUTE FOR MATERIALS RESEARCH conducts materials research leading to improved methods of measurement standards, and data on the properties of well-characterized materials needed by industry, commerce, educational institutions, and Government; develops, produces, and distributes standard reference materials; relates the physical and chemical properties of materials to their behavior and their interaction with their environments; and provides advisory and research services to other Government agencies. The Institute consists of an Office of Standard Reference Materials and the following divisions:

Analytical Chemistry—Polymers—Metallurgy—Inorganic Materials—Physical Chemistry.

THE INSTITUTE FOR APPLIED TECHNOLOGY provides technical services to promote the use of available technology and to facilitate technological innovation in industry and Government, cooperates with public and private organizations in the development of technological standards, and test methodologies; and provides advisory and research services for Federal, state, and local government agencies. The Institute consists of the following technical divisions and offices:

Engineering Standards—Weights and Measures—Invention and Innovation—Vehicle Systems Research—Product Evaluation—Building Research—Instrument Shops—Measurement Engineering—Electronic Technology—Technical Analysis.

THE CENTER FOR RADIATION RESEARCH engages in research, measurement, and application of radiation to the solution of Bureau mission problems and the problems of other agencies and institutions. The Center consists of the following divisions:

Reactor Radiation—Linac Radiation—Nuclear Radiation—Applied Radiation.

THE CENTER FOR COMPUTER SCIENCES AND TECHNOLOGY conducts research and provides technical services designed to aid Government agencies in the selection, acquisition, and effective use of automatic data processing equipment; and serves as the principal focus for the development of Federal standards for automatic data processing equipment, techniques, and computer languages. The Center consists of the following offices and divisions:

Information Processing Standards—Computer Information—Computer Services—Systems Development—Information Processing Technology.

THE OFFICE FOR INFORMATION PROGRAMS promotes optimum dissemination and accessibility of scientific information generated within NBS and other agencies of the Federal Government; promotes the development of the National Standard Reference Data System and a system of information analysis centers dealing with the broader aspects of the National Measurement System, and provides appropriate services to ensure that the NBS staff has optimum accessibility to the scientific information of the world. The Office consists of the following organizational units:

Office of Standard Reference Data—Clearinghouse for Federal Scientific and Technical Information [3]—Office of Technical Information and Publications—Library—Office of Public Information—Office of International Relations.

[1] Headquarters and Laboratories at Gaithersburg, Maryland, unless otherwise noted; mailing address Washington, D.C. 20234.
[2] Located at Boulder, Colorado 80302.
[3] Located at 5285 Port Royal Road, Springfield, Virginia 22151.

U.S. DEPARTMENT OF COMMERCE
Maurice H. Stans, Secretary

NATIONAL BUREAU OF STANDARDS
Lewis M. Branscomb, Director

Order all publications from the Superintendent of Documents
U.S. Government Printing Office, Washington, D.C. 20402.

Use of funds for printing this publication approved by the Director of the Bureau of the Budget (June 22, 1966).

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 74B, No. 2, April–June 1970

Acoustic Propagation and Stability Within an Inviscid, Heat-Conducting Fluid

J. E. McKinney and H. J. Oser

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

(January 26, 1970)

Propagation of acoustic waves within a continuum of inviscid, compressible, heat-conducting fluid is evaluated in detail in terms of both frequency (steady-state) and time dependent (transient) functions. The analysis reveals that when the value of the ratio of specific heats, γ, lies between one and two, the apparent steady-state solutions are conjugate to unstable, or regenerative, "transient" solutions, and, thus, are unacceptable. Propagation is stable for other values ($\gamma = 1, \gamma \geq 2$). The common assumption that the steady-state phase velocity varies continuously with increasing frequency from adiabatic to isothermal values is shown to be invalid, except when $\gamma = 2$.

Key words: Absorption; acoustic; adiabatic; fluid; heat-conducting; inviscid; isothermal; Kirchoff-Langevin equation; stability; steady-state; thermal; transient.

1. Introduction

This paper presents an analysis of the propagation of acoustic waves of small amplitude within a continuum of inviscid, compressible, heat-conducting fluid. The analysis gives the conditions imposed on such a fluid for the propagation to be stable. In this usage "stable" propagation is that manifested by a "stable" fluid, which has no internal energy sources (i.e., a passive fluid). With acoustic propagation the concept of stable propagation implies that for a medium subjected to sinusoidally time-dependent excitation commencing at some time, the response at any point in the medium will approach a bounded steady state. Although the requirement for stable propagation is obvious for any physically realistic fluid (as far as we know), its full significance has apparently been overlooked. In general the inclusion of this requirement will impose a relationship [1][1] between certain thermophysical properties of the fluid.

In an earlier paper [1] stability criteria, given as inequalities in terms of certain thermophysical properties, were developed for viscous, heat-conducting fluids, which are a more general class. These criteria were developed without directly analyzing the explicit time dependence of the exact solution. In the following presentation the inviscid fluid, which is simpler to analyze, is examined in detail with respect to both steady-state and transient functions. The analysis reveals some contradictions in present generally accepted concepts with respect to propagation in inviscid fluids. Propagation in these fluids has been assumed to be stable and the steady-state propagation functions have been assumed to vary continuously from adiabatic conditions at low frequencies to isothermal at high frequencies. Our analysis reveals that propagation is unstable when the value of the ratio of specific heats, γ, lies between one and two, and continuous variation in the steady-state propagation functions from adiabatic conditions at low frequencies to isothermal at high frequencies is not possible in any stable fluid, except when $\gamma = 2$. The unstable class of fluids may, of course, be stabilized [1] by the inclusion of viscosity of sufficient magnitude within the model.

Considerable discussion on propagation in heat-conducting fluids has appeared in the literature over the past 100 years, for which some of the more recent examples are given in references [2–9].

[1] Figures in brackets indicate the literature references at the end of this paper.

List of Symbols

$a = \rho c_S^2 C_V/\kappa = C_V/\kappa B_S$

$b = \rho c_S^2/(\upsilon + 4\eta/3)$

$B^* = B' - iB'' =$ complex compressibility

$B' =$ Re B^*

$B'' = -$ Im B^*

$B_S =$ adiabatic (isentropic) compressibility

$B_T = \gamma B_S =$ isothermal compressibility

$B(t) =$ transient compressibility

$c =$ phase velocity (frequency dependent)

$c_S =$ adiabatic (isentropic) phase velocity

$C_P =$ specific heat at constant pressure

$C_V =$ specific heat at constant volume

$G^* =$ complex shear modulus

$H_V =$ retardation spectrum for volume deformation

$i =$ imaginary unit

$J_n =$ Bessel Function

$K^* = K' + iK'' =$ complex bulk modulus

$M^* = M' + iM'' =$ complex longitudinal wave modulus

$p = s + i\omega =$ complex angular frequency

$p_0 = s_0 + i\omega_0 =$ singular point of $B^*(p)$

$\bar{p}_0 = s_0 - i\omega_0 =$ complex conjugate of p_0

$s =$ Re p

$s_0 =$ Re p_0

$t =$ time

$u =$ displacement

$x =$ position coordinate

$\alpha =$ Re Γ

$\beta = \omega/c =$ Im Γ

$\Gamma = \alpha + i\beta =$ complex propagation function

$\gamma = C_P/C_V = B_T/B_S =$ ratio of specific heats, or compressibilities

$\eta =$ shear viscosity

$\kappa =$ thermal conductivity

$\rho =$ density

$\tau =$ time, relaxation time

$\upsilon =$ dilatational viscosity

$\omega =$ Im $p =$ angular frequency

$\omega_0 =$ Im p_0

$\mathfrak{B}^*(p) =$ normalized dispersion function pertaining to complex compressibility

$\mathfrak{B}(t) = \mathscr{L}^{-1}[\mathfrak{B}^*(p)/p]$

$\mathscr{L} =$ Laplace transform

$\mathscr{L}^{-1} =$ inverse Laplace transform

As a result of the complicated algebraic form of the apparent steady-state solutions and the difficulties which arise in obtaining the general solution [9] of the differential equation, the invalid assumptions mentioned in the last paragraph have been widely accepted. In many cases conclusions were based largely on qualitative considerations (for example, references [10–13]) and approximations which facilitate simple calculations, but which shroud other important information. Recently, the analysis has been facilitated by the advent of fast digital computers, which make it a practical matter to trace the branches of the exact propagation functions over a wide frequency range for as many different cases as necessary to illustrate the complete behavior.

2. Steady-State Functions

2.1. Propagation Functions

In this work we are principally concerned with the forced vibration (sinusoidal) solutions of the differential equation [4] (see appendix),

$$\frac{\partial}{\partial t}\left[\frac{\partial^2 u}{\partial t^2}-c_S^2\frac{\partial^2 u}{\partial x^2}\right]=\frac{\kappa}{\rho C_V}\frac{\partial^2}{\partial x^2}\left[\frac{\partial^2 u}{\partial t^2}-\frac{c_S^2}{\gamma}\frac{\partial^2 u}{\partial x^2}\right], \tag{1}$$

where $u=u(x, t)$ is the displacement [2] of an element in the x direction at time t, c_S is the adiabatic (or isentropic) phase velocity, κ is the thermal conductivity, ρ is the density, taken here in the absence of any disturbance within the medium, C_V is the specific heat at constant volume, and γ is the ratio of specific heats (C_P/C_V). c_S, κ, ρ, C_V, and γ are taken to be constants of the fluid independent of x and t, which implies that these quantities are independent of the small fluctuations appropriate to the linearized equation (eq (1)). Equation (1) describes the infinitesimal (i.e., linear), uniaxial response of a hypothetical continuum of compressible, inviscid, heat-conducting fluid. This is probably the simplest case for which the role of thermal conductivity is considered in acoustic propagation. For our purposes eq (1) may be taken axiomatically and is used as a point of departure for subsequent arguments.

We are concerned with obtaining the exact solution [3] of eq (1) subjected to the boundary condition $u(0, t)=u_0U(t)\cos\omega t$ (along with other obvious statements with respect to initial conditions and behavior as $x\to\infty$) where $U(t)$ is the unit step function. Although obtaining the Laplace transform, $\mathscr{L}[u(x, t)]=\bar{u}(x, p)$, is straightforward procedure, obtaining its inverse $u(x, t)$ appears to be extremely difficult. However, sufficient information can be obtained from $\bar{u}(x, p)$ to reveal that the exact solution is unbounded, or unstable, when $1<\gamma<2$.

In view of the difficulties in obtaining the exact solution, we will assume the form of the steady-state solution of eq (1). From this we will obtain the frequency-dependent propagation function $\Gamma(p)$ and its equivalent response function $B^*(p)$, which appears in simpler form. The stability will be evaluated by applying Laplace transform theory to the response function, and verified later by obtaining the corresponding time-dependent (transient) function.

For waves traveling in the positive x direction only with sinusoidal time dependence, the apparent [4] "steady-state" solutions of eq (1) may be expressed in the following form:

$$u(x, t)=\mathrm{Re}\,[Ae^{pt-\Gamma x}]_{s=0}=Ae^{-\alpha x}\cos(\omega t-\beta x) \tag{2}$$

where $p=s+i\omega$ is the complex angular frequency,[5] and $\Gamma=\alpha+i\beta$ is the complex (frequency dependent) propagation function with $\beta=\omega/c$, where c is the frequency dependent phase velocity.

---

[2] The equation for velocity at a point, $v=v(x, t)$, has identical form.

[3] This type (including transient response) of analysis with respect to acoustic propagation is, in general, fairly recent in the literature. For the exact solution pertaining to a viscous, nonconducting fluid, see, for example, reference [14].

[4] We speak of eq (2) as an "apparent" steady-state solution inasmuch as it appears to be a steady state and satisfies the differential equation (eq (1)). However, it remains to be seen if eq (2) is the exact solution as $t\to\infty$.

[5] The complex frequency, $p=s+i\omega$, is used for the purpose of applying the Laplace transform in the subsequent discussions. It is convenient to use the form $\exp(pt-\Gamma x)$ in lieu of $\exp(\Gamma x-pt)$ in order to facilitate the subsequent use of this transform without modification.

A is a constant (for each branch), which is taken to be real without loss in generality.

Substitution of eq (2) into eq (1) gives the following frequency dependence of the complex propagation function $\Gamma(p)$:

$$\Gamma^2(p)=\frac{\gamma p^2}{2c_S^2}\left[1+\frac{a}{p}+\left(1+\frac{2(\gamma-2)a}{\gamma p}+\frac{a^2}{p^2}\right)^{1/2}\right] \tag{3}$$

where $a=\rho c_S^2 C_V/\kappa$, or alternatively, $a=C_V/\kappa B_S$, remembering $c_S^2=1/\rho B_S$ where B_S is the adiabatic compressibility. Equation (3) may be identified as the propagation function for "steady-state" solution of the Kirchhoff-Langevin equation [8, 9] with zero viscosity, or equivalently, with an infinite viscous Reynolds number.

For waves with diminishing intensity in the direction of propagation, as required for passive fluids, α and β must have the same sign. This requirement is met for all four branches given by eq (3). The two branches for which α and β are both negative are ignored in this analysis because they correspond to backward traveling waves and contribute no additional information to the properties of the fluid. The two branches implied by the $\frac{1}{2}$-power term in eq (3) are commonly called the acoustic and thermal branches. As customary, the acoustic branch is taken to be the one for which the absorption, α, vanishes in the low-frequency limit in contrast to an unbounded value in the thermal branch. This choice is consistent with the acoustic branch propagating adiabatically in the low-frequency limit.

Expressions for α and β as functions of real frequency are obtained tediously, producing expressions containing eight radicals each, which, indeed, exceed the practical limit of comprehension and space available here. However, the values of these functions with frequency were computed for several examples and are illustrated in section 2.4 along with their corresponding complex response functions.

2.2. Complex Response Functions

For mathematical convenience the propagation function obtained in the last section is converted to the complex response function which describes the response to alternating pressure of any small volume element of constant mass within the acoustic field. The information obtained in these two representations is equivalent.

The longitudinal wave propagation function and its corresponding complex response functions are related [15] by

$$M^*=K^*+\frac{4}{3}G^*=\rho p^2/\Gamma^2 \tag{4}$$

where $M^*=M'+iM''$ is commonly called the longitudinal wave modulus, which is appropriate to describe linear response for uniaxial strain with sinusoidal time dependence. K^* and G^* are the complex bulk and shear moduli, respectively. Since we are concerned with inviscid, compressible fluids, which support no shear stress, G^* may be taken to be identically zero from which follows

$$B^*=\Gamma^2/\rho p^2, \tag{5}$$

where $B^*=B'-iB''=1/K^*$ is the complex compressibility. Thus, B^* and Γ^2 have the same frequency dependence except for a factor of p^2. The alternative expressions[6] equivalent to eq (5) in terms of real and imaginary components at real frequencies are

$$\begin{aligned} B' &= (\beta^2-\alpha^2)/\omega^2\rho \\ B'' &= 2\alpha\beta/\omega^2\rho. \end{aligned} \tag{6}$$

[6] From inspection of eq (6) it is clear that there are only two distinct branches of $B^*(p)$ from the four branches of $\Gamma(p)$ given by eq (3), since α and β always have the same sign.

As commonly used in acoustics (and in this discussion) the complex compressibility relates the dilatation to pressure in both amplitude and phase for sinusoidally time dependent excitation. $B^*(p)$ is not a property of the fluid alone, as are the isothermal and adiabatic compressibilities for a viscous fluid, but may also depend on the boundary conditions. Other response functions, such as the characteristic impedance, or admittance, of the medium, could be analyzed in this manner. Since these response functions are equivalent, the conditions for stability obtained through analysis of any of them remain the same. The complex compressibility appears to be the most convenient function for studying this particular case.

The identity,

$$\sqrt{c+id}=\sqrt{\frac{1}{2}\left(c+\sqrt{c^2+d^2}\right)}+i\,\frac{d}{|d|}\sqrt{\frac{1}{2}\left(-c+\sqrt{c^2+d^2}\right)} \tag{7}$$

where c and d may take on all real values, is useful for establishing the inverse relations to eq (6) and certain other useful relations obtained subsequently. In this paper the notation $z^{1/2}$ is used for the double-valued function. Expressions of the form $\sqrt{z}$ are interpreted to be single-valued functions with a nonnegative real part. The other branch will be denoted by $-\sqrt{z}$ (with a nonpositive real part). This convention so far does not distinguish the branches of $z^{1/2}$ when the value of z is on the negative real axis. For a reason which will become apparent in the treatment of the special case $\gamma=2$ (sec. 4.2), the branch cut is taken along the negative real axis and is assigned the polar argument the value $\theta=-\pi$ (instead of the more customary choice $\theta=\pi$). Thus, for the first branch $(-\pi\leq\theta<\pi)$ we write $z^{1/2}=\sqrt{z}$, which has a nonnegative real part, and for the second branch $(\pi\leq\theta<3\pi)$, $z^{1/2}=-\sqrt{z}$, with a nonpositive real part. This choice has the consequence that $\sqrt{-1}=-i$ instead of $+i$ as usual. The form $\pm\sqrt{z}$ or $\mp\sqrt{z}$ is used according to the convention that the upper sign applies to the acoustic branch, whenever applicable.

The inverse relations to eqs (6),

$$\begin{aligned}\alpha&=\omega\left[\frac{\rho}{2}\left(-B'+\sqrt{B'^2+B''^2}\right)\right]^{1/2}\\ \beta&=\omega\left[\frac{\rho}{2}\left(B'+\sqrt{B'^2+B''^2}\right)\right]^{1/2},\end{aligned} \tag{8}$$

are readily obtained from eqs (5) and (7).

From eqs (3) and (5) the explicit frequency dependence of the complex compressibility is

$$B^*(p)=\frac{\gamma B_S}{2}\left[1+\frac{a}{p}+\left(1+\frac{2(\gamma-2)a}{\gamma p}+\frac{a^2}{p^2}\right)^{\frac{1}{2}}\right] \tag{9}$$

which, after taking $p=i\omega$ $(s=0)$, becomes

$$B^*(\omega)=\frac{\gamma B_S}{2}\left[1-\frac{ia}{\omega}\pm\sqrt{1-\frac{a^2}{\omega^2}-\frac{2i(\gamma-2)a}{\omega}}\right],\ 1\leq\gamma<2 \tag{10a}$$

$$B^*(\omega)=\frac{\gamma B_S}{2}\left[1-\frac{ia}{\omega}\mp\sqrt{1-\frac{a^2}{\omega^2}-\frac{2i(\gamma-2)a}{\omega}}\right],\ \gamma\geq 2. \tag{10b}$$

The following real and imaginary components of $B^*(\omega)=B'-iB''$ are obtained by applying eq (7):

$$B'(\omega)=\frac{\gamma B_S}{2}\left\{1\pm\sqrt{\frac{1}{2}\left[1-\frac{a^2}{\omega^2}+\sqrt{\left(1-\frac{a^2}{\omega^2}\right)^2+\left(\frac{2(\gamma-2)a}{\omega\gamma}\right)^2}\right]}\right\},\ 1\leq\gamma<2 \tag{11a}$$

$$B'(\omega)=\frac{\gamma B_S}{2}\left\{1\mp\sqrt{\frac{1}{2}\left[1-\frac{a^2}{\omega^2}+\sqrt{\left(1-\frac{a^2}{\omega^2}\right)^2+\left(\frac{2(\gamma-2)a}{\omega\gamma}\right)^2}\right]}\right\},\ \gamma\geq 2 \tag{11b}$$

$$B''(\omega)=\frac{\gamma B_S}{2}\left\{\frac{a}{\omega}\mp\sqrt{\frac{1}{2}\left[-\left(1-\frac{a^2}{\omega^2}\right)+\sqrt{\left(1-\frac{a^2}{\omega^2}\right)^2+\left(\frac{2(\gamma-2)a}{\omega\gamma}\right)^2}\right]}\right\},\ \gamma\geq 1 \tag{11c}$$

Equations (10) and (11) are in keeping with our convention given in the last paragraph and consistent with our previously stated choice (sec. 2.1) that the acoustic branch be the one for which the absorption vanish in the limit $\omega\to 0$. This choice is consistent with $B^*(0)=B_S$ for the acoustic branch. The two expressions of $B^*(\omega)$ and $B'(\omega)$ are shown here for the purpose of distinguishing between the acoustic (upper sign) and thermal (lower sign) branches. The reason for the sign reversal at $\gamma=2$ is found in the fact that the roles of the two branches are interchanged in a way which should become clearer during further discussions. Values of γ less than unity are not considered because these would require the existence of a negative compressibility or specific heat. It will be shown in the next section that for $1<\gamma<2$, the apparent steady-state solutions are misleading because the fluids are unstable in that the corresponding "transient" functions are regenerative in time. Thus, in excluding the unstable solutions, eqs (10a) and (11a) are applicable only to the special case where $\gamma=1$.

From initial inspection eq (9) appears to have a simple pole at the origin in addition to the branch-point singularities. A binomial expansion of the radical reveals that the simple pole in the acoustic branch disappears by virtue of a cancellation in terms of a/p. The presence of the simple pole in the thermal branch is consistent with the unbounded absorption in the low-frequency limit.

2.3. Evaluation of Stability from the Location of Singularities

It is commonly assumed in wave propagation that transient solutions decaying with time, are conjugate to steady-state solutions decaying in the direction of propagation. In this section it will be shown that this assumption is not correct in general and is violated for the response functions under consideration when $1<\gamma<2$.

Since the general solution of eq (1) is extremely difficult and apparently has not been obtained, we have chosen an alternate approach, namely, to study the transient [7] response function $B(t)$, where $B(t)$ is the ratio of time-dependent dilatation, to step function pressure. $B(t)$ corresponds to the response function $B^*(p)$ in frequency space. (See eq (9).) The complex and transient functions are related by

$$B^*(p)=p\mathscr{L}[B(t)]$$

with the corresponding inverse

$$B(t)=\mathscr{L}^{-1}[B^*(p)/p], \tag{12}$$

where $\mathscr{L}$ and $\mathscr{L}^{-1}$ are the Laplace and inverse Laplace transform operators. We say a solution of eq (1) is stable if the corresponding transient response function is bounded for all times t after removal of terms linear in t. This requires $B^*(p)$ to have singularities only on, or to the left of, the imaginary p axis. (See, for example, [16, 17].) Terms of $B(t)$ linear in t (corresponding to simple poles of $B^*(p)$ at the origin) are permitted, although they are unbounded in the limit as $t\to\infty$. These terms are analogous to those for steady-flow shear within a Newtonian fluid. In general, for compliances (resulting from step-function stress) of the form ct^n, the only acceptable values of n for stable response are $0\leq n\leq 1$. $n<0$ would give a singularity at $t=0$ implying an infinite instantaneous deformation from the application of stress. $n>1$ characterizes a deformation which would continue to increase after removal of the stress (and without bound in the limit $t\to\infty$), as

[7] The response to step-function excitation is commonly called the transient response function, in view of the fact that this function usually vanishes at large times. In this case the so-called "transient" response function $B(t)$ often grows with time and is unbounded in the limit as $t\to\infty$. This behavior is contrary to that implied by the ordinary definition of transient, i.e., existing over only a short time.

can be seen by applying the Boltzmann superposition principle. $0 < n < 1$ corresponds to a deformation which would recover completely on removal of stress but would increase without limit as long as the stress is applied. $n=0$ and $n=1$ represent purely elastic and purely viscous response, respectively.

In addition to the simple pole at the origin, branch point singularities appear at the values

$$p=-\frac{(\gamma-2)a}{\gamma}\pm\frac{2ia\sqrt{\gamma-1}}{\gamma}.$$

Thus, as required for real time-dependent response functions, singularities off the real axis are always complex conjugates. The locus of these singularities with respect to γ is a circle obeying the relation

$$s_0^2+\omega_0^2=a^2. \tag{13}$$

This plot is shown on figure 1 in which the values of γ are given for the intercepts. Singularities

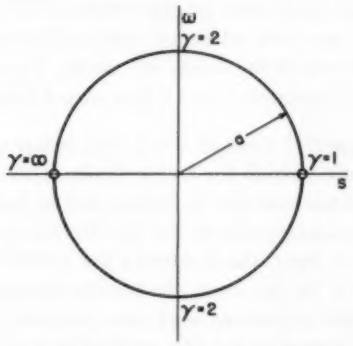

FIGURE 1. *Locus of the branch-point singularities of the complex response function,* $B^*(p)$, *given by eq* (9).

Open circles corresponding to $\gamma=1$ and $\gamma=\infty$ indicate exclusions from the locus.

are restricted to be on and to the left of the imaginary axis when $\gamma \geq 2$. When γ is exactly one, the radical in eq (9) is a complete square and the point at $p=a$ is regular. However, as soon as γ differs from unity (even infinitesimally), a pair of branch points appears and the corresponding instability sets in. Accordingly, the point $\gamma=1$ is an isolated stable point. At $\gamma=\infty$ the branch-point singularities become either simple poles or vanish entirely. (See sec. 4.3.) Accordingly, the points $p=a$ (corresponding to $\gamma=1$) and $p=-a$ (corresponding to $\gamma=\infty$) are excluded from the locus as indicated by the open circles in figure 1. Since the singularities are paired symmetrically about the real p axis, the transient response will be oscillatory. (See sec. 3.2.) When the singularities are paired about the negative real axis as with $\gamma \geq 2$, the oscillations damp out with time. When the singularities are paired about the positive real axis, as they appear for $1 < \gamma < 2$, the oscillations amplify with time and are unbounded as $t \to \infty$.

2.4. Graphical Display of the Functions

The values of the apparent steady-state functions plotted in this section were obtained in three different ways for the purpose of checking the validity of the results. The functions were programmed in OMNITAB and also in FORTRAN using both the real and complex modes. With OMNITAB and FORTRAN using the real mode, eqs (11) were used to compute the values of the real and imaginary components of the normalized complex response functions. These were converted to the values of the normalized propagation functions using eqs (8) with $\beta=\omega/c$. With FORTRAN using the complex mode eqs (10) were used to compute the values of the normalized complex response function, which were converted to the values of the normalized complex propaga-

tion function using eq (5) rewritten in the form

$$\Gamma(\omega) = -\omega\sqrt{\rho B^*(\omega)}\,.$$

Since we have taken the branch cut of the function $z^{1/2}$ in a manner consistent with $\sqrt{-1} = -i$ (see sec. 2.2.), whereas the FORTRAN complex mode assigns the value i to the operation $\sqrt{-1}$, it was necessary to include a statement to make the corresponding inversion at $\gamma = 2$ over the range $\omega > a$ (see sec. 4.2.). Figures 2 and 3 give the computed dimensionless complex response functions plotted against dimensionless frequency over the interesting range including the value corresponding to $\omega = a$ for different values of γ. The principal distinction between these two figures is that figures 2 and 3 pertain to unstable and stable fluids, respectively. Figures 4 and 5 give the corresponding dimensionless propagation functions, which are, of course, more directly realizable experimentally and discussed in the literature. The functions for which the fluids are unstable are included because they are illustrations of functions which have been considered to be stable. Figures 2 and 3 with $\gamma = 1.5$ are an example in the range of $1 < \gamma < 2$ over which the fluids are unstable. The unstable fluids are the only ones (except when $\gamma = 2$) which would seem to propagate monotonically and continuously between adiabatic and isothermal limits. Figures 3 and 5 with $\gamma = 1$, 2, and 2.5 are examples for which the fluids are stable. From these figures the nature of the coalescence and inversion of the branches at $\gamma = 2$ (see sec. 4.2) is revealed more clearly.

With the exception of the special case of $\gamma = 1$, the compressibility always vanishes in the high-frequency limit of the acoustic branch for stable fluids. This corresponds to so-called "anomalous" propagation in that the phase velocity is unbounded in the high-frequency limit. Although the acoustic branch propagates adiabatically in the low-frequency limit, and the thermal branch, isothermally in the high-frequency limit, the functions are never continuous between these limits for stable fluids except when $\gamma = 2$. In this case, the infinite derivatives at the value corresponding to $\omega = a$ in all of the functions are consistent with the existence of a branch-point singularity at this value. This singularity also accounts for the coalescence of $B'(\omega)$ and $B''(\omega)$ in the range $\omega \leqslant a$ and $\omega \geqslant a$, respectively, with $\gamma = 2$.

The dotted lines shown with the case for $\gamma = 2.5$ were calculated for the components of the viscoelastic response function

$$B^*(i\omega) = \frac{B_S}{1 + i\omega\tau} \tag{14}$$

where τ is the relaxation [8] time for a single element relaxation spectrum. The limiting values of eq (14) for small and large p are the same as those for eqs (11b) and (11c) using the upper signs. A single relaxation manifests the narrowest dispersion region possible with respect to any nonnegative relaxation distribution. Accordingly, it is clear from inspection alone of figures 3c and 5c that the inviscid functions under discussion cannot be represented by any distribution of relaxation (or retardation) times in the form

$$B^*(p) = B_g + \int_0^\infty \frac{H_V(\tau)}{1 + p\tau}\, d\tau \tag{15}$$

with $H_V(\tau)$ nonnegative. For a single relaxation time $H_V(\tau)$ is a delta function multiplied by a constant. Since eq (15) applies only to functions with singularities on the negative real p axis, it is not applicable, in general, to the functions under discussion.

[8] This usage is apparently common in acoustic nomenclature. According to viscoelastic nomenclature [18], this quantity should be called a retardation time when used in connection with compliances as with eqs (14) and (15) in contrast to a relaxation time, with moduli.

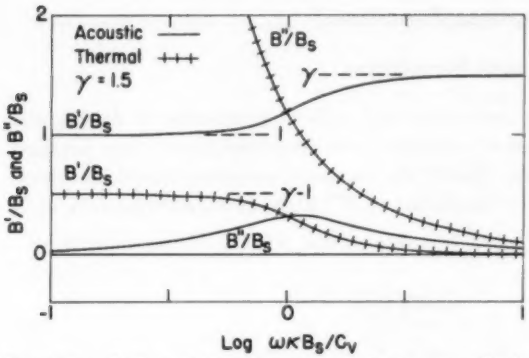

FIGURE 2. *Real and imaginary parts of the normalized complex response functions as functions of normalized frequency with* $\gamma = 1.5$ *as an example in the range of* $1 < \gamma < 2$ *for which the propagation is unstable.*

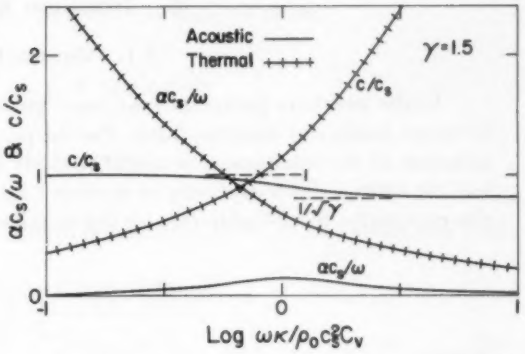

FIGURE 4. *Normalized propagation functions corresponding to the functions in figure 2 (unstable propagation).*

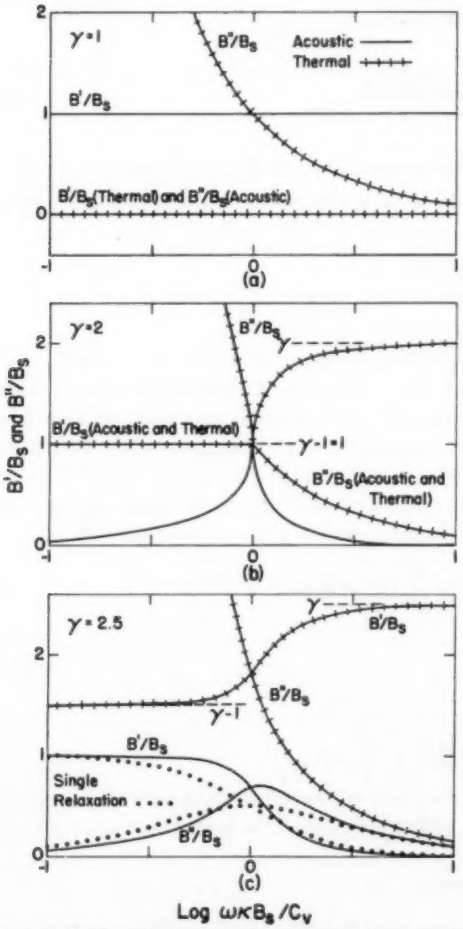

FIGURE 3. *Real and imaginary parts of the normalized complex response functions as functions of normalized frequency with* (a) $\gamma = 1$, (b) $\gamma = 2$, *and* (c) $\gamma = 2.5$ *as examples at* $\gamma = 1$, *and in the range of* $\gamma \geq 2$ *for which the propagation is stable.*

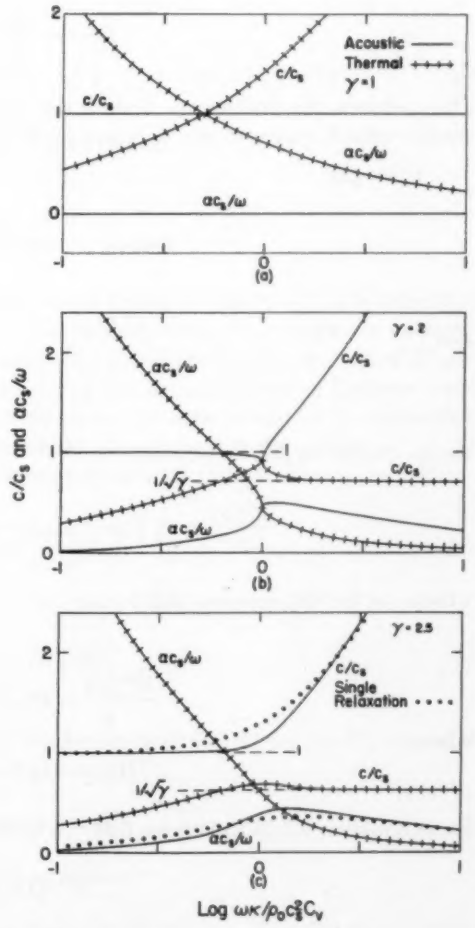

FIGURE 5. *Normalized propagation functions corresponding to the functions in figure 3 (stable propagation).*

3. Transient Response Functions [9]

3.1. General Integral Formulation

In the previous paragraphs we have given a necessary and sufficient criterion to distinguish between stable and unstable fluids. For the purpose of obtaining additional insight into the detailed behavior of the solutions, it is useful to study more closely the transient response functions which can be written down explicitly as inverse Laplace transforms of $B^*(p)/p$. It is convenient to split the expression for $B^*(p)/p$ (see eq (9)) as follows:

$$\frac{B^*(p)}{p} = \frac{\gamma B_S}{2}\left[\frac{1}{p} + \frac{a}{p^2} + \frac{\mathfrak{B}^*(p)}{p}\right], \tag{16}$$

where

$$\frac{\mathfrak{B}^*(p)}{p} = \frac{1}{p}\left[1 + \frac{2(\gamma-2)a}{\gamma p} + \frac{a^2}{p^2}\right]^{\frac{1}{2}}. \tag{17}$$

Application of the inverse Laplace transform to eq (16) yields:

$$B(t) = \frac{\gamma B_S}{2}\left[1 + at + \mathfrak{B}(t)\right]. \tag{18}$$

This reduces the problem to finding the explicit time dependence of $\mathfrak{B}(t)$, a function which is double valued, owing to the ½-power in $\mathfrak{B}^*(p)$.

If we put

$$s_0 = \frac{(2-\gamma)a}{\gamma}$$

$$\omega_0 = \frac{2a\sqrt{\gamma-1}}{\gamma}$$

and

$$p_0 = s_0 + i\omega_0$$

$$\bar{p}_0 = s_0 - i\omega_0,$$

we may write eq (17) as follows:

$$\frac{\mathfrak{B}^*(p)}{p} = \frac{1}{p^2}\left[(p-p_0)(p-\bar{p}_0)\right]^{1/2} = \frac{1}{p^2}\left[p^2 - 2s_0 p + a^2\right]^{1/2}, \tag{19}$$

which can be decomposed still further into

$$\frac{\mathfrak{B}^*(p)}{p} = D(p) - \frac{2s_0}{p}D(p) + \frac{a^2}{p^2}D(p),$$

where

$$D(p) = \left[p^2 - 2s_0 p + a^2\right]^{-1/2}.$$

From equation 5.3.34 in [19] we find

$$\mathscr{L}^{-1}\{D(p)\} = e^{s_0 t} J_0(\omega_0 t).$$

[9] See footnote 7.

Writing $\pm$ to allow for the double-valuedness of the ½-power function, we obtain

$$\pm \mathfrak{B}(t) = e^{s_0 t} J_0(\omega_0 t) - 2s_0 \int_0^t e^{s_0 \tau} J_0(\omega_0 \tau) d\tau + a^2 \int_0^t \int_0^{t'} e^{s_0 \tau} J_0(\omega_0 \tau) d\tau dt'.$$

Partial integration in the double integral and application of equation 11.3.5 in [20] yields the final result:

$$\pm \mathfrak{B}(t) = e^{s_0 t}[(1 - s_0 t) J_0(\omega_0 t) - \omega_0 t J_1(\omega_0 t)] + (a^2 t - s_0) \int_0^t e^{s_0 \tau} J_0(\omega_0 \tau) d\tau. \tag{20}$$

Remembering our convention that the upper sign pertains to the acoustic branch, the exact form of the transient response function $B(t)$ for $\gamma \geq 2 (s_0 \leq 0)$ is obtained from substituting eq (20) into eq (18):

$$B(t) = \frac{\gamma B_S}{2} \left\{ 1 + at \mp \left[e^{s_0 t} (1 - s_0 t) J_0(\omega_0 t) - \omega_0 t J_1(\omega_0 t) + (a^2 t - s_0) \int_0^t e^{s_0 \tau} J_0(\omega_0 \tau) d\tau \right] \right\}, \quad \gamma \geq 2. \tag{21}$$

When $1 \leq \gamma < 2$ $(s_0 \geq 0)$, eq (21) is valid also except $\mp$ becomes $\pm$.

3.2. Approximate Behavior for Large t

Although eq (21) is the general relation applicable for all values of t and $\gamma \geq 1$, it is difficult to obtain comprehensible information with respect to the explicit asymptotic behavior as $t \to \infty$ which is of particular interest here. To obtain this information we take the expansions of $B^*(p)/p$ about the three singularities $p = 0$, $p = p_0$, and $p = \bar{p}_0$. (From the general theory of Laplace transforms [21] the asymptotic behavior of $y(t)$for large t may be readily determined by the behavior of the leading terms of $\mathscr{L}[y(t)] = y^*(p)/p$ expanded about the singularities in the p plane.)

In this connection we are interested in evaluating eq (16) (sec. 3.1) which is

$$\frac{B^*(p)}{p} = \frac{\gamma B_S}{2} \left[\frac{1}{p} + \frac{a}{p^2} + \frac{\mathfrak{B}^*(p)}{p} \right] \tag{16}$$

where

$$\frac{\mathfrak{B}^*(p)}{p} = \frac{1}{p} \left[1 + \frac{2(\gamma - 2)}{\gamma p} + \frac{a^2}{p^2} \right]^{1/2}. \tag{17}$$

Since the inverse transforms of the first two terms in the brackets of eq (16) are readily available, we are only concerned with the appropriate expansions of eq (17).

a. Expansion about $p = 0$

For the expansion about $p = 0$, we rewrite eq (17) as follows:

$$\frac{\mathfrak{B}^*(p)}{p} = \frac{a}{p^2} \left(1 + \frac{2(\gamma - 2)p}{a\gamma} + \frac{p^2}{a^2} \right)^{1/2}.$$

For small p the above equation may be expanded by the binomial theorem indicating double valuedness by the $\mp$ sign.

$$\frac{\mathfrak{B}^*(p)}{p} = \mp\left(\frac{a}{p^2} + \frac{\gamma - 2}{\gamma p} + \ldots\right).$$

The dots indicate regular terms which do not contribute to the asymptotic expansions of either $B(t)$ or $\mathfrak{B}(t)$. The corresponding expression for $\mathfrak{B}(t)$, for large t, is then

$$\mathfrak{B}_0(t) \approx \mp\left(at + \frac{\gamma - 2}{\gamma}\right),$$

where the subscript 0 indicates that only the contribution from $p = 0$ is included.

b. Expansion about $p = p_0$

In this case we rewrite eq (17) as

$$\frac{\mathfrak{B}^*(p)}{p} = \frac{a}{p^2}\left[(p - p_0)(p - \bar{p}_0)\right]^{1/2}$$

which when rewritten in appropriate form for expansion about $p = p_0$ becomes

$$\frac{\mathfrak{B}^*(p)}{p} = \left[\frac{a(p_0 - \bar{p}_0)^{1/2}}{p_0^2} \frac{\left(1 + \dfrac{p - p_0}{p_0 - \bar{p}_0}\right)^{1/2}}{\left(1 + \dfrac{p - p_0}{p_0}\right)^2}\right](p - p_0)^{1/2},$$

where all the terms within the brackets are regular at $p = p_0$. Consequently,

$$\frac{\mathfrak{B}^*(p)}{p} = \frac{a(p_0 - \bar{p}_0)^{1/2}}{p_0^2}(p - p_0)^{1/2}(1 + \ldots),$$

where the omitted terms in the right-hand parenthesis are ignored because they give rise to a series expansion of $\mathfrak{B}(t)$ in decreasing powers of t, thus contributing less as $t \to \infty$. From reference [21] we find

$$\mathfrak{B}_1(t) \approx \mp \frac{a}{2p_0^2}\sqrt{\frac{p_0 - \bar{p}_0}{\pi}}\, e^{p_0 t} t^{-3/2}, \qquad t \to \infty$$

as the leading term for the contribution from $p = p_0$.

c. Expansion about $p = \bar{p}_0$

The expansion about $p = \bar{p}_0$ is obtained from the above expression for $\mathfrak{B}_1(t)$ by interchanging p_0 and $\bar{p}_0$, which is tantamount to taking the complex conjugate of $\mathfrak{B}_1(t)$. Thus,

$$\overline{\mathfrak{B}}_1(t) \approx \mp \frac{a}{2\bar{p}_0^2}\sqrt{\frac{\bar{p}_0 - p_0}{\pi}}\, e^{-\bar{p}_0 t}\, t^{-3/2}, \qquad t \to \infty$$

pertains to $p = \bar{p}_0$.

d. Summation of Contributions

Adding the above contributions ($\mathfrak{B}_0$, $\mathfrak{B}_1$, and $\overline{\mathfrak{B}}_1$) together along with the contributions from the two left-hand terms in eq (16), we obtain the following general ($\gamma \geq 1$) asymptotic expression for $B(t)$ as $t \to \infty$:

$$B(t) \approx \frac{\gamma B_S}{2}\left\{1 + at \mp \left[at + \frac{\gamma - 2}{\gamma} + \frac{a}{\sqrt{\pi}}\, t^{-3/2}\, \mathrm{Re}\left(\frac{\sqrt{p_0 - \bar{p}_0}}{p_0^2}\, e^{p_0 t}\right)\right]\right\}.$$

Alternatively, the following equivalent expression may be obtained in terms of real quantities:

$$B(t) \approx \frac{\gamma B_S}{2}\left\{1 + at \mp \left[at + \frac{\gamma - 2}{\gamma} + \frac{a}{(s_0^2 + \omega_0^2)^2}\sqrt{\frac{\omega_0}{\pi}}\left((s_0^2 - \omega_0^2 + 2s_0\omega_0)\cos \omega_0 t - (s_0^2 - \omega_0^2 - 2s_0\omega_0)\sin \omega_0 t\right) t^{-3/2} e^{s_0 t}\right]\right\}, \qquad (22)$$

where the upper sign now pertains to the acoustic branch for all $\gamma \geq 1$.

For the stable class $\gamma \geq 2$ the asymptotic behavior is

$$B(t) \approx \frac{\gamma B_S}{2}\left[1 + at \mp \left(at + \frac{\gamma - 2}{\gamma}\right)\right].$$

Asymptotes linear in t (although unbounded in the limit $t \to \infty$) are consistent with stable solutions for the reason given in section 2.3. At large t sinusoidal oscillations about these asymptotes are damped by the factor $t^{-3/2}e^{s_0 t}$ for which $s_0 \leq 0$. For the unstable class $1 < \gamma < 2$ these oscillations regenerate by the same factor for which $s_0 > 0$.

4. Special Cases for Different Values of γ

The following discussion includes the special cases of the functions for different values or ranges of γ. In these cases the functions may reduce to considerably simpler form and may also reveal distinct characteristics not immediately obvious from the more general form.

4.1. Stable Case $\gamma = 1$

When $\gamma = 1$, the reduced expression for eq (9) becomes

$$B^*(p) = \frac{B_S}{2}\left[1 + \frac{a}{p} \pm \left(1 - \frac{a}{p}\right)\right]; \gamma = 1. \qquad (23)$$

Thus, it is apparent that $B' = B_S$ and $B'' = 0$ for the acoustic branch, and $B' = 0$ and $B'' = C_V/\kappa\omega$ for the thermal branch. These values are consistent with a constant adiabatic (or isothermal) phase velocity with no absorption in the acoustic branch. The thermal branch propagates in a manner analogous to shear propagation within a Newtonian fluid [2] in that

$$\alpha = \beta = \sqrt{\frac{\omega\rho C_V}{2\kappa}}.$$

The steady-state functions for this case are displayed in figures 3a and 5a.

At $\gamma = 1$ the branch point singularities vanish because the square in eq (19) is completed. Accordingly, this point is an isolated stable point and the response is characterized by the simple pole at the origin.

The following expression for $B(t)$ (which is exact for all t) is obtained by operating on eq (23) directly:

$$B(t)=\frac{B_S}{2}\left[1+at\pm(1-at)\right],\quad \gamma=1.$$

It is apparent from the above equation that when $\gamma=1$, the thermal branch (lower sign) of $B(t)$ is analogous to the shear compliance of a Newtonian fluid subjected to a constant stress. Transient terms linear in t (although unbounded in the limit $t\to\infty$) are consistent with stable solutions for the reason given in section 2.3.

4.2. Stable Case $\gamma=2$

When $\gamma=2$, eqs (11b) and (11c) reduce to

$$B'(\omega)=B_S\left\{1\mp\sqrt{\frac{1}{2}\left[1-\frac{a^2}{\omega}+\left|1-\frac{a^2}{\omega^2}\right|\right]}\right\},$$

$$B''(\omega)=B_S\left\{\frac{a}{\omega}\mp\sqrt{\frac{1}{2}\left[-\left(1-\frac{a^2}{\omega^2}\right)+\left|1-\frac{a^2}{\omega^2}\right|\right]}\right\},\quad \gamma=2.$$

It is apparent from the above equations that B' is single valued and constant over the range $\omega\leq a$, and B'' is single valued over the range $\omega\geq a$, in a manner consistent with the branch point singularities at $p=\pm ia$. In this case the two branches of B' coalesce for $0\leq\omega\leq a$ by virtue of the fact that $(\gamma-1)B_S=B_S$. The steady-state functions for this case are displayed in figures 3b and 5b.

Since this case corresponds to $d=0$ in eq (7), and because of the single-valuedness mentioned above we have some liberty in designating the branches with respect to acoustic or thermal over the range $\omega>a$. If we had used the customary convention in setting $\sqrt{-1}=i$, in lieu of $\sqrt{-1}=-i$ (see discussion after eq (7), sec. 2.2), the branches (as shown on fig. 3b) would be interchanged over the range $\omega>a$. Since this case has been shown to be stable, we have chosen to deviate from the customary convention. This results in the limiting behavior of $B^*(\infty)$ being the same as that for the stable class $\gamma>2$ in that

$$\text{Acoustic}\qquad B^*(\infty)=0,\quad \gamma\geq 2$$

$$\text{Thermal}\qquad B^*(\infty)=\gamma B_S,\quad 1<\gamma<2.$$

The response function $B(t)$ valid for all t (expressed in integral form), and approximate for large t, is obtained by setting $s_0=0$ in eqs (21) and (22), respectively. The asymptotes are

$$\lim_{t\to\infty}B(t)=B_S[1+at\mp at],\qquad \gamma=0.$$

At large t the sinusoidal oscillations about these asymptotes are damped by the factor $t^{-3/2}$. This behavior, which is characterized by the branch points, is distinct from that of undamped resonance, characterized by simple poles also paired symmetrically on the imaginary p axis.

4.3. Stable Case $\gamma=\infty$

Since there is no phenomenological bound on the maximum value of γ, we are considering the role of all possible values for which $1\leq\gamma\leq\infty$. (For example, γ is apparently singular at the critical point [22] in that C_P is more singular than C_V or alternatively, B_T is more singular than B_S.) The general form of $B^*(p)$ taken from eq (9) in the limit as $\gamma\to\infty$ is

$$\lim_{\gamma\to\infty}B^*(p)=\frac{B_T}{2}\left\{1+\frac{a}{p}\mp\left[1+\frac{a}{p}-O\left(\frac{2a}{\gamma(p+a)}\right)\right]\right\}. \tag{24}$$

where the O symbol is the one used in asymptotic theory. The value of $B^*(p)$ at $\gamma=\infty$ depends on the dependence of B_S on γ in the limit as $\gamma\to\infty$. The behavior of $B^*(p)$ at $\gamma=\infty$ is summarized in table 1, where the acoustic and thermal branches are treated separately. It is interesting to observe from comparison of eqs (9) and (24) that the branch-point singularities coalesce into a simple pole at $p=-a$ corresponding to $\gamma=\infty$, and, as indicated by table 1, the simple pole at $p=-a$ vanishes for finite values of B_T ($B_S=0$).

The general form of $B(t)$ in the limit as $\gamma\to\infty$ corresponding to eq (24) is

$$\lim_{\gamma\to\infty} B(t)=\frac{B_T}{2}\left\{1+at\mp\left[1+at-\mathrm{O}\left(\frac{2}{\gamma}\left(1-e^{-at}\right)\right)\right]\right\}.$$

The behavior of $B(t)$ at $\gamma=\infty$, is also summarized in table 1. Although $B(t)$ may be unbounded at $\gamma=\infty$ for all values of t under certain conditions as shown, this case is considered to be a stable one because the response does not grow with time after removal of the terms linear in time. A medium of infinite compressibility, for example, a vacuum, does not, necessarily, have intrinsic energy sources as implied by our criterion for an unstable fluid.

TABLE 1. *Behavior of* B*(p) *and* B(t) *at* $\gamma=\infty$

$\lim_{\gamma\to\infty} B_S(\gamma)$	$B^*(p)$, Acoustic	$B^*(p)$, Thermal	$B(t)$, Acoustic	$B(t)$, Thermal
$B_S=0$	0	$B_T(1+a/p)$	0	$B_T(1+at)$
$B_S=$ Const	$aB_S/(p+a)$	∞	$B_S(1-e^{-at})$	∞
$B_S=(\text{Const})^{\gamma\gamma}$, $\epsilon>0$	∞	∞	∞	∞

5. Summary of Asymptotic Values and Relations Between Asymptotics of $B^*(p)$ and $B(t)$

Table 2 summarizes the asymptotics for $B^*(p)$ and $B(t)$. The limiting values for $B^*(p)$ with $p=0$, and $p\to\infty$ are taken from the appropriate binomial expansions of eq (9). The values of $B(t)$ with $t=0$, and $t\to\infty$ are taken from eqs (21) and (22), respectively. The $\pm(\infty)$ notation indicates regenerative oscillations modulated symmetrically about the value zero with the amplitudes going to infinity as $t\to\infty$.

The following give the relations between the asymptotes of $B^*(p)$ and $B(t)$

$$\begin{aligned} B^*(\infty) &= B(0), && \gamma\geq 1\\ B^*(0) &= B(\infty), && \gamma\geq 2,\ \gamma=1\\ B^*(0) &\neq B(\infty), && 1<\gamma<2 \end{aligned}$$

The first is readily obtained from Laplace transform theory without recourse to any relations given in this paper. The first two are always obeyed in passive response functions. The reason the second is not obeyed in the unstable class ($1<\gamma<2$) is simply because the rightmost singularities appear to the right of the imaginary axis of the p plane.

TABLE 2. *Summary of asymptotics of* B*(p) *and* B(t)

	Stable class ($\gamma=1$, $\gamma\geq 2$)		Unstable class ($1<\gamma<2$)	
	Acoustic	Thermal	Acoustic	Thermal
$B^*(0)$	B_S	$B_S(\gamma-1+\gamma a/p)$	B_S	$B_S(\gamma-1+\gamma a/p)$
$B(\infty)$	B_S	$B_S(\gamma-1+\gamma at)$	$B_S\pm(\infty)$	$B_S(\gamma-1+\gamma at)\pm(\infty)$
$B^*(\infty)$	0	B_T	B_T	0
$B(0)$	0	B_T	B_T	0

For meaning of $\pm(\infty)$, see text.

6. Concluding Remarks

The analysis of the differential equation for acoustic propagation in a continuum of an inviscid, compressible, heat-conducting fluid yields solutions which display unexpected properties. Although the "apparent" steady-state, forced-vibration solutions correspond to a physically reasonable behavior in that they predict oscillations which always decay in the direction of propagation, the corresponding "transient" response functions predict a response (to step-function excitation) which at long times may oscillate with increasing amplitudes. A regenerative "transient" response implies that the exact solution of the wave equation oscillates with increasing amplitude which is unbounded in the limit of infinite time, and, accordingly, a true steady-state solution does not exist. Since we restrict "acceptable" solutions to those which predict physically realistic behavior for passive, or stable, fluids, we reject the regenerative, or unstable, solutions.

The following statements are considered by the authors to be the most important consequences of the foregoing analysis:

1. Contrary to the popular concept, the phase velocity can never vary continuously with frequency from adiabatic to isothermal values in a compressible, inviscid, heat-conducting fluid, except when the ratio of specific heats is exactly equal to two.

2. Propagation within the above fluid is stable only when either of the following conditions hold:

$$\text{(i)}\quad \gamma = 1$$

or

$$\text{(ii)}\quad \gamma \geq 2.$$

With the unstable class $(1 < \gamma < 2)$ the "steady-state" phase velocity of the acoustic branch appears to vary continuously with frequency from adiabatic to isothermal values. With this class the apparent steady-state solution is actually conjugate to a regenerative "transient" solution unbounded in the limit as $t \to \infty$.

With fluids where the ratio of specific heats lies between the values one and two, stability may be achieved by the inclusion of a minimum viscosity. In a previous paper [1] the stability criteria were developed in terms of thermophysical properties applicable to all values of γ in this more general case. These requirements are summarized as follows:

$$\gamma = 1$$

or

$$\frac{(2-\gamma)\kappa}{\left(\upsilon + \frac{4}{3}\eta\right) C_P} \leq 1, \qquad \gamma > 1$$

where υ and η are the bulk and shear viscosities. Either of the above conditions is sufficient to insure stability, and these two are the only stable cases.

Although the inviscid fluid is a hypothetical one (since all fluids presumably exhibit a finite viscosity), the unstable fluids cannot be stabilized by the inclusion of a viscosity of infinitesimal magnitude. The results from reference [1] clearly show that a lower bound exists on viscosity to insure stability when $1 < \gamma < 2$. The above stability requirements were found to be obeyed for all the fluids checked except the molten metals, including mercury, and possibly the superfluids. These results indicate that in some cases different assumptions would have to be made in deriving the propagation equation (Kirchhoff-Langevin equation) for compressible, viscous, heat-conducting fluids.

In general an alternative evaluation of stability may be done by an appropriate qualitative analysis using the Kronig-Kramers [23], or the equivalent Ginzberg [24], relations, which relate the real and imaginary parts of passive complex functions. (The Ginzberg relations apply specifically to velocity and absorption.) In this manner it is often possible to relate the sign of one complex component

to the sign of the trend of the other. For example, referring to figures 2 and 3c, it is possible to show that B' (acoustic) must be monotonically decreasing, and B' (thermal), monotonically increasing, with frequency for stable propagation. These trends are not obeyed in figure 2 because corresponding propagation would be unstable. The results with respect to stability are equivalent to those from the foregoing analysis because the Kronig-Kramers relations imply the existence of all singularities on, or to the left of, the imaginary axis of the complex p plane.

The results of this work and those from the more general case [1] indicate the possibility that the apparent steady-state relations obtained from either phenomenological or molecular theories may be inconsistent with stable propagation. In particular, whenever thermoconductivity is included in a model, stability may be questionable. As illustrated here, the fact that the apparent steady-state solutions predict diminishing intensity in the direction of propagation is necessary, but not sufficient, for stable propagation.

7. Appendix. Derivation of the Propagation Equation

Equation (1) of the text may be obtained from the hydrodynamic equations [25] applicable to a Newtonian fluid under very general conditions. These equations are the equation of continuity, the Navier-Stokes equation, and the Fourier-Kirchhoff-Neumann energy equation. After dropping viscosity terms and linearizing (replacing D/Dt by $\partial/\partial t$, dropping products of derivatives, and taking ρ constant when it appears in a product with a quantity which goes to zero with vainishing motion), these equations for motion in the x direction are

$$\frac{\partial \rho}{\partial t}+\rho \frac{\partial v}{\partial x}=0 \tag{A.1}$$

$$\rho \frac{\partial v}{\partial t}=-\frac{\partial P}{\partial x} \tag{A.2}$$

$$\rho \frac{\partial e}{\partial t}=-P\frac{\partial v}{\partial x}+\kappa \frac{\partial^2 \theta}{\partial x^2}, \tag{A.3}$$

where v, θ, P, and e are the velocity, temperature, pressure, and internal energy per unit mass, respectively.

For an inviscid fluid we take $e=e(\rho, P)$ and $\theta=\theta(\rho, P)$ to be single-valued functions of the indicated arguments. From eqs (A.2) and (A.3) the following ordinary thermodynamic relations are implied:

$$\left(\frac{\partial e}{\partial P}\right)_\rho=\frac{\gamma B_S C_V}{\alpha P}$$

$$\left(\frac{\partial e}{\partial \rho}\right)_P=\frac{p}{\rho^2}-\frac{\gamma C_V}{\rho \alpha_P}.$$

Expanding eq (A.3) by the chain rule, we obtain

$$\rho\left(\frac{\partial e}{\partial \rho}\right)_P \frac{\partial \rho}{\partial t}+\rho\left(\frac{\partial e}{\partial P}\right)_\rho \frac{\partial P}{\partial t}=-P\frac{\partial v}{\partial x}+\kappa\left(\frac{\partial \theta}{\partial \rho}\right)_P \frac{\partial^2 \rho}{\partial x^2}+\kappa\left(\frac{\partial \theta}{\partial P}\right)_\rho \frac{\partial^2 P}{\partial x^2}$$

in which the derivatives operating on e are removed by substituting the above thermodynamic relations, and those on θ, by substituting the definitions

$$B_S=\frac{1}{\rho}\left(\frac{\partial \rho}{\partial P}\right)_S=\frac{1}{\gamma \rho}\left(\frac{\partial \rho}{\partial P}\right)_\theta, \qquad \alpha_P=-\frac{1}{\rho}\left(\frac{\partial \rho}{\partial \theta}\right)_P.$$

382-233 O - 70 - 2

Differentiation of this result with respect to x and t and substitution of eqs (A.1) and (A.2) to replace $\partial P/\partial x$ and $\partial \rho/\partial t$ gives

$$\frac{\partial}{\partial t}\left[\frac{\partial^2 v}{\partial t^2}-c_s^2\frac{\partial^2 v}{\partial x^2}\right]=\frac{\kappa}{\rho C_V}\frac{\partial^2}{\partial x^2}\left[\frac{\partial^2 v}{\partial t^2}-\frac{c_S^2}{\gamma}\frac{\partial^2 v}{\partial x^2}\right] \tag{A.4}$$

Equation (1) of the text may be obtained by substituting $\partial u/\partial t$ for v. One time derivative may be integrated out by taking a definite integral from $-\infty$ to t and setting $u(x, -\infty)$ and all of its derivatives equal to zero. The result (eq (1) of text) is simply a substitution of u for v in eq (A.4).

8. References

[1] McKinney, J. E., and Oser, H. J., J. Acous. Soc. Am. **47**, 781 (1970).
[2] Thurston, R. N., in Physical Acoustics, Vol. **I-A**, Chap. 1, W. P. Mason, Ed. (Academic Press, New York, 1964).
[3] Lindsay, R. B., Mechanical Radiation, p. 266 (McGraw-Hill Book Co., New York 1960).
[4] Vigoureux, P., Ultrasonics, p. 32 (John Wiley & Sons, New York, 1951).
[5] DeGroot, S. R., and Mazur, P., Non-Equilibrium Thermodynamics, Chap. 12 (Interscience Publishers, New York, 1962).
[6] Perepechko, I. I., Soviet Phys.-Acoustics **10**, 284 (1965).
[7] Herzfeld, K. F., and Litovitz, T. A., Absorption and Dispersion of Ultrasonic Waves, Chap. 1 (Academic Press, New York, 1959).
[8] Greenspan, M., in Physical Acoustics, Vol. **II-A**, Chap. 1, W. P. Mason, Ed. (Academic Press, New York, 1965).
[9] Truesdell, C., J. Rational Mech. Analysis **2**, 643 (1953).
[10] Condon, E. U., Am. Phys. Teacher **1**, 18 (1933).
[11] Herzfeld, K. F., and Rice, F. O., Phys. Rev. **31**, 691 (1928).
[12] Marvin, R. S., and McKinney, J. E., in Physical Acoustics, Vol. **II-B**, p. 165, W. P. Mason, Ed. (Academic Press, New York, 1965).
[13] Rayleigh, J. W. S., Theory of Sound, Vol. **II**, p. 28 (Dover, New York, 1945).
[14] Norwood, F. P., J. Acoust. Soc. Am. **44**, 450 (1967).
[15] Herzfeld, K. F. and Litovitz, T. A., Absorption and Dispersion of Ultrasonic Waves, Chap. 1, p. 451 (Academic Press, New York, 1959).
[16] Tuttle, D. F., Jr., Network Synthesis, Chap. 4 (John Wiley & Sons, New York, 1958).
[17] van der Pol, B., and Bremer, H., Operational Calculus, 167–170 (Cambridge, 1959).
[18] Leaderman, H., Trans. Soc. Rheol. **1**, 213 (1957).
[19] Bateman, H., Tables of Integral Transforms, Bateman Manuscript Project, California Institute of Technology, Vol. **I**, A. Erdelyi, Ed. (McGraw-Hill Book Co., New York, 1954).
[20] Luke, Y. L., in Handbook of Mathematical Functions, M. Abramowitz and I. A. Stegun, Eds., National Bureau of Standards Applied Mathematics Series 55 (U.S. Government Printing Office, Washington, D.C. 20402, 1964).
[21] Doetsch, F., Handbuch der Laplace Transformation, Vol. **II**, Chap. 7 (Birkhäuser Verlag, Basel, 1955).
[22] Egelstaff, P. A., An Introduction to the Liquid State, pp. 159–161 (Academic Press, New York, 1967).
[23] Van Vleck, J. H., in Properties of Short Radio Waves, p. 641, D. E. Kerr, Ed. (McGraw-Hill, New York, 1951).
[24] Ginzberg, V. L., Soviet Phys. Acoustics **1**, 32 (1957).
[25] Langlois, W. E., Slow Viscous Flow, p. 58 (MacMillan, New York, 1964).

(Paper 74B2–320)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 74B, No. 2, April–June 1970

On Some Indefinite Integrals of Confluent Hypergeometric Functions*

Edward W. Ng and Murray Geller****

(January 5, 1970)

Analytical expressions and reduction formulas are developed for various indefinite integrals of the confluent hypergeometric functions. These integrals are of the type $\int f(a, b, z)z^p e^{\alpha z}dz$, where f is one of the two Kummer functions $M(a, b, z) = {}_1F_1(a; b; z)$ or $U(a, b, z)$, with real or complex a, b, z and α, and real p.

Keys words: Bessel functions; confluent hypergeometric functions; electronic energies; indefinite integrals.

1. Introduction

Multiple integrals occur in many physical problems in the form where one of the integrations results in a higher mathematical function. Thus one is often faced with the task of evaluating integrals involving higher functions. Whereas there exists a large body of literature for integrals with limits $(0, \infty)$ and $(-\infty, \infty)$, there are considerably fewer sources pertaining to indefinite integrals. This latter type is important, for example, in statistical mechanics for the integration of a distribution over a finite portion of velocity or momentum space and in electromagnetic theory for the integration over a finite charge configuration. We note the appearance of only one major monograph with emphasis on indefinite and finite integrals of higher functions, that of Y. L. Luke [1][1] on Integrals of Bessel Functions. This monograph covers a much wider range of material than its title suggests.

In this note we investigate some indefinite integrals involving confluent hypergeometric functions of the type $\int f(a, b, z)z^p e^{\alpha z}dz$, where f is one of the two Kummer functions $M(a, b, z) = {}_1F_1(a; b; z)$ or $U(a, b, z)$. Several recent applications of these integrals to studies on molecular electronic energies [2, 3] have been made.

It is well known that the confluent hypergeometric functions contain many other higher functions as special cases. For the definitions and an extensive analysis of the properties of these functions, the reader is referred to the Handbook of Mathematical Functions [4] and the work by Slater [5].

We define the integrals to be considered in this note as follows: [2]

$$M_p(\alpha; a, b, z) \equiv \int M(a, b, z)z^p e^{\alpha z}dz, \tag{1a}$$

$$U_p(\alpha; a, b, z) \equiv \int U(a, b, z)z^p e^{\alpha z}dz, \tag{1b}$$

and the special cases

$$\Phi_n(a, b, z) \equiv M_n(0; a, b, z), \tag{2a}$$

$$\Psi_n(a, b, z) \equiv U_n(0; a, b, z). \tag{2b}$$

*An invited paper. This paper presents the results of one phase of research carried out at the Jet Propulsion Laboratory, California Institute of Technology, under Contract NAS7-100, sponsored by the National Aeronautics and Space Administration. Acknowledgment is made to Dr. M. Saffren for helpful discussions.

**Present address: Jet Propulsion Laboratory, California Institute of Technology, Pasadena, Calif. 91103.

[1] Figures in brackets indicate the literature references at the end of this paper.

[2] Throughout this paper we omit the constant for all indefinite integrals.

Here a, b, z, and α may be real or complex, p is real and n is a positive integer or 0. In section 2 we discuss the function Φ_n and in section 3 the function Ψ_n showing how both of these functions can be reduced to simple linear combinations of elementary functions and various hypergeometric functions. In section 4 we consider eq (1) with $\alpha=0$, and we conclude in section 5 with a brief discussion of the general integrals $M_p(\alpha;\, a,\, b,\, z)$ and $U_p(\alpha;\, a,\, b,\, z)$ of eq (1), relating them to some recent work by Babister [6]. In an appendix we specialize the parameters in $M_p(\alpha;\, a,\, b,\, z)$ and $U_p(\alpha;\, a,\, b,\, z)$ to the Bessel function case, $b=2a$, and, using the results of sections 4 and 5, display several interesting indefinite integrals for the Bessel functions.

2. Analysis of $\Phi_n(a,\, b,\, z)$

In this section we analyze the function $\Phi_n(a,\, b,\, z)$ and show its reduction to elementary functions or to linear combinations of various hypergeometric functions. We use the following well known properties of the $M(a,\, b,\, z)$ [4, 5]

$$(1+a-b)M(a,\, b,\, z)-aM(a+1,\, b,\, z)+(b-1)M(a,\, b-1,\, z)=0, \tag{3}$$

$$bM(a,\, b,\, z)-bM(a-1,\, b,\, z)-zM(a,\, b+1,\, z)=0, \tag{4}$$

$$\frac{d}{dz}M(a,\, b,\, z)=\frac{a}{b}M(a+1,\, b+1,\, z); \tag{5}$$

and their equivalent integrated expressions (see footnote 2)

$$(1+a-b)\Phi_n(a,\, b,\, z)-a\Phi_n(a+1,\, b,\, z)+(b-1)\Phi_n(a,\, b-1,\, z)=0, \tag{3a}$$

$$b\Phi_n(a,\, b,\, z)-b\Phi_n(a-1,\, b,\, z)-\Phi_{n+1}(a,\, b+1,\, z)=0, \tag{4a}$$

$$z^nM(a,\, b,\, z)-n\Phi_{n-1}(a,\, b,\, z)=\frac{a}{b}\Phi_n(a+1,\, b+1,\, z), \tag{5a}$$

$$M(a,\, b,\, z)=\frac{a}{b}\Phi_0(a+1,\, b+1,\, z). \tag{5b}$$

Equations (5a) and (5b) are obtained by integrating eq (5) by parts. We also note the useful expressions

$$M(0,\, b,\, z)=1, \tag{6}$$

$$\Phi_n(0,\, b,\, z)=\frac{z^{n+1}}{n+1} \tag{6a}$$

Four different cases arise dependent on whether a is equal or unequal to an integer and b is equal or unequal to an integer.

(i) $a \neq$ integer, $b \neq$ integer

Use of eq (5a) with a replaced by $a-1$ and b replaced by $b-1$ yields the reduction formula

$$\Phi_n(a,\, b,\, z)=\frac{(b-1)}{(a-1)}z^nM(a-1,\, b-1,\, z)-n\frac{(b-1)}{(a-1)}\Phi_{n-1}(a-1,\, b-1,\, z). \tag{7}$$

Repeated application of eq (7) generates $\Phi_n(a,\, b,\, z)$ as a linear combination of terms

$z^{n-n'}M(a-n'-1,\ b-n'-1,\ z)$, with n' ranging from 0 to $n-1$, plus the lowest order term, $\Phi_0(a-n,\ b-n,\ z)$, which is given by eq (5b).

(ii) $a \neq$ integer, $b =$ integer > 0

We first note that the integral, as well as $M(a, b, z)$, is undefined for $b = -m$. For $b \neq 1$ we can replace $(b-1)\Phi_{n-1}(a-1, b-1, z)$ in eq (7) by its equivalent from eq (3a) with a replaced by $a-1$ and n replaced by $n-1$, to yield a reduction formula analogous to eq (7)

$$\Phi_n(a, b, z) = \frac{(b-1)}{(a-1)} z^n M(a-1, b-1, z) - n\Phi_{n-1}(a, b, z) + n\frac{(a-b)}{(a-1)}\Phi_{n-1}(a-1, b, z). \quad (8)$$

Repeated application of eq (8) generates for Φ_n a linear combination of M's plus the lowest order terms $\Phi_0(a', b', z)$ which we obtain from eq (5b). For the case $b = 1$ we see that eq (8) is still valid if we recognize from eqs (3) and (4) that

$$\lim_{b \to 1} (b-1)M(a-1, b-1, z) = (a-1)zM(a, 2, z). \quad (9)$$

Substituting eq (9) into eq (8), we have the reduction formula

$$\Phi_n(a, 1, z) = z^{n+1}M(a, 2, z) - n[\Phi_{n-1}(a, 1, z) - \Phi_{n-1}(a-1, 1, z)], \quad (10)$$

with the lowest order term being given by

$$\Phi_0(a, 1, z) = zM(a, 2, z). \quad (11)$$

One can easily show by mathematical induction that the general result for the case $b = 1$ is

$$\Phi_n(a, 1, z) = z^{n+1} \sum_{m=0}^{n} (-1)^m \binom{n}{m} \frac{1}{(m+1)} M(a, m+2, z) \quad (12)$$

which holds equally well when a is an integer.

(iii) $a =$ integer, $b \neq$ integer

For $a > n+1$, eq (7) can be used to reduce $\Phi_n(a, b, z)$ to a series of M's plus the lowest order term $\Phi_0(a-n, b-n, z)$ given by eq (5b). For $a = n+1$ and $a = n > 0$, repeated application of eq (7) again reduces Φ_n to a series of M's plus the lowest order terms $\Phi_0(1, b-n, z)$ and $\Phi_1(1, b-n+1, z)$ respectively. These are related through eq (4a) with $a = 1$ and $n = 0$ and eq (6a) by

$$\Phi_1(1, b-n+1, z) = (b-n)\ [\Phi_0(1, b-n, z) - z]. \quad (13)$$

The primitive function $\Phi_0(1, b', z)$, not reducible any further, can be written [1, p. 5]

$$\Phi_0(1, b', z) = z_2F_2(1, 1; 2, b'; z). \quad (14)$$

For the special case where b' is an integer, the ${}_2F_2$ function can be reduced to incomplete gamma functions and exponential integrals as will be seen in case (iv). The condition $n > a > 0$ is easily handled by repeated use of eq (4a) with n replaced by $n-1$ and b replaced by $b-1$

$$\Phi_n(a, b, z) = (b-1)\ [\Phi_{n-1}(a, b-1, z) - \Phi_{n-1}(a-1, b-1, z)\ ], \quad (15)$$

until the reduction generates either of the lowest order terms $\Phi_{n'}(0, b, z)$ given by eq (6a), $\Phi_0(a' \neq 1, b', z)$ of eq (5b), or $\Phi_0(1, b', z)$, the primitive given in eq (14). Lastly, for the case $a = -m$, $m = 0, 1, 2, \ldots$, since $M(-m, b, z)$ is a polynomial of $m+1$ terms we can either employ eq (5a) backwards until we arrive at $\Phi_{n+m}(0, b+m, z)$, given by eq (6a), or directly integrate to obtain

$$\Phi_n(-m, b, z) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{z^{n+k+1}}{(b)_k (n+k+1)}, \tag{16}$$

where

$$(b)_k = \Gamma(b+k)/\Gamma(b).$$

(iv) a = integer, b = integer > 0.

With $a \geq b$, repeated use of eq (4) generates

$$M(b+m, b, z) = e^z \sum_{k=0}^{m} \binom{m}{k} \frac{z^k}{(b)_k}, \tag{17}$$

since

$$M(b, b, z) = e^z, \tag{18}$$

which holds even when $b \neq$ integer. Integration of eq (17) obviously yields

$$\Phi_n(b+m, b, z) = \sum_{k=0}^{m} \binom{m}{k} \frac{1}{(b)_k} \int z^{n+k} e^z dz. \tag{19}$$

For the case $b > a > 0$, we have from eq (3a) with $b = a+m$, the recursion relation

$$(1-m)\Phi_n(a, a+m, z) = a\Phi_n(a+1, a+m, z) - (a+m-1)\Phi_n(a, a+m-1, z). \tag{20}$$

Equation (20) is repeatedly applied until the right hand side contains only terms of the form $b = a+1$. Using eq (4a) with b replaced by a and n replaced by $n-1$, we have the reduction formula

$$\Phi_n(a, a+1, z) = a \int z^{n-1} e^z dz - a\Phi_{n-1}(a-1, a, z). \tag{21}$$

The reduction by eq (21) is continued, generating a series of terms $\int z^k e^z dz$ plus the lowest order term. If $a \leq n$, the lowest order term is $\Phi_{n-a}(0, 1, z)$, which is given by eq (6a). With $a = n+1$, the lowest order term is

$$\Phi_0(1, 2, z) = \int (e^z - 1) \frac{dz}{z}, \tag{22}$$

or equivalently

$$\Phi_0(1, 2, x) = Ei(x) - \ln x \qquad x > 0, \tag{23a}$$

$$\Phi_0(1, 2, -z) = -E_1(z) - \ln z \qquad |\arg z| < \pi, \tag{23b}$$

where E_1 and Ei are the exponential integrals [4, Chap. 5]. Finally, if $a > n+1$, the lowest order term is $\Phi_0(a-n, a-n+1, z)$. Using eq (5b), we can express this in terms of $M(a-n-1, a-n, z)$. Since

$$M(n, n+1, -x) = \frac{n!}{x^n} [1 - e^{-x} e_{n-1}(x)], \tag{24}$$

where $e_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$; we have

$$\Phi_0(a-n, a-n+1, z) = (-1)^{a-n-1} \frac{(a-n)(a-n-2)!}{z^{a-n-1}} [1 - e^z e_{a-n-2}(-z)]. \tag{25}$$

We note that since

$$\gamma(a, x) = a^{-1}x^a e^{-x} M(1, 1+a, x) = a^{-1}x^a M(a, 1+a, -x), \tag{26}$$

where $\gamma(a, x)$ is the incomplete gamma function [7], the $\Phi_0(1, b', z)$ of eq (14) can alternatively be written as

$$\Phi_0(1, b', z) = (b'-1)\int \gamma(b'-1, z) z^{1-b'} e^z dz, \tag{27}$$

whereas $\Phi_0(a, 1+a, -x)$ can be expressed as

$$\Phi_0(a, 1+a, -x) = a\int \gamma(a, x) x^{-a} dx, \tag{28a}$$

For $a \neq 1$, this reduces to

$$\Phi_0(a, a+1, -x) = \frac{a}{1-a}[x^{1-a}\gamma(a, x) + e^{-x}]. \tag{28b}$$

For $a = 1$, eq. (28a) reduces to eq. (23b), and with $a =$ integer > 1, eq (28b) reduces to eq (25).

The last case to be considered in this section is $b > a$ where $b > 0$ and $a = -m$. This case is immediately obtainable from eq. (16) since that equation obviously holds for $b =$ integer > 0.

3. Analysis of $\Psi_n(a, b, z)$

Analogous to the results of section 2, the function $\Psi_n(a, b, z)$ can be reduced to elementary functions or to linear combinations of various hypergeometric functions. Here, too, four different cases, a equal or unequal to an integer and b equal or unequal to an integer, must be considered for the complete reduction of Ψ_n. We commence the analysis with the definition of $U(a, b, z)$ [5]

$$U(a, b, z) = \frac{\pi}{\sin \pi b}\left[\frac{M(a, b, z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b}\frac{M(1+a-b, 2-b, z)}{\Gamma(a)\Gamma(2-b)}\right], \tag{29}$$

and the well known properties [4, 5]

$$U(a, b, z) - aU(a+1, b, z) - U(a, b-1, z) = 0, \tag{30}$$

$$(b-a)U(a, b, z) + U(a-1, b, z) - zU(a, b+1, z) = 0, \tag{31}$$

$$\frac{d}{dz}U(a, b, z) = -aU(a+1, b+1, z). \tag{32}$$

We also note the integrated form of eqs (30)–(32)[2]

$$\Psi_n(a, b, z) - a\Psi_n(a+1, b, z) - \Psi_n(a, b-1, z) = 0, \tag{30a}$$

$$(b-a)\Psi_n(a, b, z) + \Psi_n(a-1, b, z) - \Psi_{n+1}(a, b+1, z) = 0, \tag{31a}$$

$$z^n U(a, b, z) - n\Psi_{n-1}(a, b, z) = -a\Psi_n(a+1, b+1, z), \tag{32a}$$

$$U(a, b, z) = -a\Psi_0(a+1, b+1, z). \tag{32b}$$

Equation (32b) is simply obtained from eq (32a) by setting $n = 0$. Other properties of the $U(a, b, z)$ used in this analysis are

$$U(0, b, z) = 1, \tag{33}$$

$$\Psi_n(0, b, z) = z^{n+1}/(n+1), \tag{33a}$$

$$U(a, a+1, z) = z^{-a}, \tag{34}$$

$$U(a, 1-n, z) = z^n U(a+n, 1+n, z), \tag{35}$$

the last equation being a special case of Kummer's transformation. One final property of the $U(a, b, z)$, as opposed to the $M(a, b, z)$, is that they are defined for all values of a and b.

(i) $a \neq$ integer, $b \neq$ integer

Using eq (32a) with a replaced by $a-1$ and b replaced by $b-1$, we have

$$\Psi_n(a, b, z) = \frac{1}{(1-a)}[z^n U(a-1, b-1, z) - n\Psi_{n-1}(a-1, b-1, z)]. \tag{36}$$

Repeated use of eq (36) reduces Ψ_n to a series of terms $z^{n-n'}U(a-n'-1, b-n'-1, z)$ plus the lowest order term, which by eq (32b) is given by

$$\Psi_0(a', b', z) = \frac{1}{(1-a')}U(a'-1, b'-1, z), \tag{37}$$

or equivalently, from eq (30) with a replaced by $a-1$

$$\Psi_0(a', b', z) = \frac{1}{(1-a')}U(a'-1, b', z) + U(a', b', z). \tag{38}$$

(ii) $a \neq$ integer, $b =$ integer

For $b \geq 1$, if we replace both terms on the right hand side of eq (36) by their equivalents from eq (30) and (30a) with a replaced by $a-1$, then we have the reduction formula for Ψ_n

$$\Psi_n(a, b, z) = [z^n\{U(a-1, b, z) + (1-a)U(a, b, z)\} - n\{\Psi_{n-1}(a-1, b, z) + (1-a)\Psi_{n-1}(a, b, z)\}]; \tag{39}$$

the lowest order term is given by eq (38). If $b \leq 0$, we use the integrated form of Kummer's transformation, eq (35), to yield

$$\Psi_n(a, -m, z) = \Psi_{n+m+1}(a+m+1, m+2, z), \tag{40}$$

which can obviously be handled by means of eqs (39) and (38) above.

(iii) $a =$ integer, $b \neq$ integer

If $a = -m$, $m = 0, 1, 2, \ldots$, then from eq (31a) with $a = -m+1$, we have the recursion relation

$$\Psi_n(-m, b, z) = \Psi_{n+1}(-m+1, b+1, z) - (b+m-1)\Psi_n(-m+1, b, z), \tag{41}$$

which reduces to $\Psi_{n'}(0, b', z)$ of eq (33a) upon repeated usage. An alternative approach is to note that since

$$U(-m, b, z) = (-1)^m m!\, L_m^{(b-1)}(z), \tag{42}$$

where $L_m^{(a)}(z)$ is the associated Laguerre function [4, chap. 22], direct integration yields

$$\Psi_n(-m, b, z) = (-1)^m m! \sum_{k=0}^{m} \binom{m+b-1}{m-k} \frac{(-1)^k}{k!} \frac{z^{n+k+1}}{(n+k+1)}. \tag{43}$$

For $a > 0$, three distinct situations arise. If $a > n+1$, use of eq (36) reduces $\Psi_n(a, b, z)$ to a series of U's plus the lowest order term $\Psi_0(a-n, b-n, z)$ given by eq (37) as in case (i) above. With $a = n$ or $a = n+1$, repeated application of eq (36) reduces Ψ_n to a series of U's plus the lowest order terms $\Psi_0(1, b-n, z)$ and $\Psi_1(1, b-n+1, z)$ respectively. These are related through eq (31a) with $n=0$ and $a=1$ by

$$\Psi_1(1, b'+1, z) = z + (b'-1)\Psi_0(1, b', z). \tag{44}$$

Furthermore, from the definition of $U(a, b, z)$, eq (29), with $a=1$, we see that

$$\Psi_0(1, b', z) = \Gamma(b'-1) \int \frac{e^z}{z^{b'-1}}\, dz - \frac{1}{(b'-1)}\, \Phi_0(1, b', z), \tag{45}$$

where the first term in eq (45) is an incomplete gamma function and the second term is precisely the primitive function discussed in section 2, eq (14). We remark that for the case $b =$ integer, eq (45) reduces to powers, logarithms and the exponential integral as will be seen in (iv) below. Finally, for $a < n$, use of eq (31a) with n replaced by $n-1$ and b replaced by $b-1$ leads to the reduction formula

$$\Psi_n(a, b, z) = (b-a-1)\Psi_{n-1}(a, b-1, z) + \Psi_{n-1}(a-1, b-1, z). \tag{46}$$

Continued application of eq (46) results in the lowest order terms $\Psi_{n'}(0, b', z)$ given by eq (33a) and $\Psi_{n'}(n', b', z)$, the case $a = n$ discussed above.

(iv) $a =$ integer, $b =$ integer

Corresponding to eq (26) for the representation of the incomplete gamma function, $\gamma(a, x)$, in terms of M, we have

$$\Gamma(a, x) = x^a e^{-x} U(1, 1+a, x) = e^{-x} U(1-a, 1-a, x). \tag{47}$$

Therefore for the case $a = b = -m$, $m = 0, 1, 2, \ldots$, we have

$$\Psi_n(-m, -m, z) = \int z^n e^z \Gamma(m+1, z)\, dz. \tag{48}$$

Replacing $\Gamma(m+1, z)$ by its expansion as a truncated exponential and integrating termwise, we obtain

$$\Psi_n(-m, -m, z) = m! \sum_{k=0}^{m} \frac{1}{k!} \frac{z^{n+k+1}}{(n+k+1)}. \tag{49}$$

For $a = b = +m$, we have a two-step reduction procedure. First, from eq (30a) with $a = m$ and $b = m+1$ we have

$$\Psi_n(m+1, m+1, z) = \frac{1}{m} \int z^{n-m} dz - \frac{1}{m}\, \Psi_n(m, m, z), \tag{50}$$

where we have used the integrated form of eq (34) to eliminate $\Psi_n(m, m+1, z)$. This reduction is continued until $\Psi_n(1, 1, z)$ is generated. Second, noting that

$$U(1, 1, z) = e^z E_1(z), \tag{51}$$

multiplying by z^n and integrating by parts, we obtain the reduction formula

$$\Psi_n(1, 1, z) = z^n e^z E_1(z) + \frac{z^n}{n} - n\Psi_{n-1}(1, 1, z). \tag{52}$$

Equation (52) is applied n times until $\Psi_0(1, 1, z)$ is generated which, by integration of eq (51), is obviously given by

$$\Psi_0(1, 1, z) = \ln z + e^z E_1(z). \tag{53}$$

This same result is obtained from eq (47) with a chosen as $1-m$ and the replacement under the integral sign of $\Gamma(1-m, x)$ by $x^{1-m}E_m(x)$.

For $b > a$, if $a \leq 0$, eq (41) is repeatedly applied until a series of terms $\Psi_{n'}(0, b', z)$ is obtained which is given by eq (33a). With $a > 0$, specifically choosing $a = m+1$ and $b = m+m'+2$ in eq (30a), we have the recursion relation

$$\Psi_n(m+1, m+m'+2, z) = (m+1)\Psi_n(m+2, m+m'+2, z) + \Psi_n(m+1, m+m'+1, z). \tag{54}$$

This recursion is repeated until the right-hand side contains only terms with $b = a+1$, given by the integral of eq (34). The general result is then simply

$$\Psi_n(m+1, m+m'+2, z) = \frac{1}{m!}\sum_{k=0}^{m'}\binom{m'}{k}(m+k)!\int z^{n-k-m-1}dz. \tag{55}$$

Finally, the case $a > b$ is either treated by the reduction formula eq (41) for $a \leq 0$ until $\Psi_{n'}(0, b', z)$ of eq (33) is obtained, or for $a > 0$, by repeated use of eq (31a) until $a = 0$ or $a = b = +m$ (considered in eqs (50)–(53) above) is reached. Often one can utilize eq (35) to reduce the parameters to lower orders before starting the recursions. As a simple example of the case $a > b$, we present the result for $\Psi_2(4, 2, z)$

$$\Psi_2(4, 2, z) = \Psi_1(3, 0, z) = \frac{z^2}{12}\left[(1+z) - z(2+z)e^z E_1(z)\right]. \tag{56}$$

4. $M_p(0; a, b, z)$ and $U_p(0; a, b, z)$

Consider first the integral $M_p(0; a, b, z)$. When the series for $M(a, b, z)$ is integrated termwise, we find that

$$M_p(0; a, b, z) = \sum_{k=0}^{\infty}\frac{(a)_k}{(b)_k}\frac{1}{k!}\frac{\Gamma(p+k+1)}{\Gamma(p+k+2)}z^{p+k+1} = \frac{z^{p+1}}{p+1}{}_2F_2(p+1, a; p+2, b; z), \tag{57}$$

provided that $b \neq 0, -1, -2, \ldots$, and $p \neq -1, -2, \ldots$. We note that for $p = -m$, with $m = 1, 2, \ldots$,

$$M_{-m}(0; a, b, z) = \frac{(a)_m}{(b)_m}\frac{z}{m!}{}_3F_3(a+m, 1, 1; b+m, 2, 1+m; z) - \sum_{k=0}^{m-2}\frac{(a)_k}{(b)_k}\frac{1}{k!}\frac{1}{(m-k-1)}z^{1+k-m} + \frac{(a)_{m-1}}{(b)_{m-1}}\frac{1}{(m-1)!}\ln z. \tag{58}$$

From the definition for $U(a, b, z)$ in terms of $M(a, b, z)$, eq (29), we have

$$U_p(0; a, b, z) = \frac{\pi}{\sin \pi b}\left[\frac{1}{\Gamma(1+a-b)\Gamma(b)}M_p(0; a, b, z) - \frac{1}{\Gamma(a)\Gamma(2-b)}M_{p+1-b}(0, 1+a-b, 2-b, z)\right]. \tag{59}$$

Contiguous relations for the generalized hypergeometric functions, ${}_rF_q$, are given by Rainville [8]. From the special cases for $r=q=2$, we obtain

$$aM_p(0;\, a+1,\, b,\, z)=z^{p+1}M(a,\, b,\, z)-(p+1-a)M_p(0;\, a,\, b,\, z), \tag{60}$$

$$(b-1)M_p(0;\, a,\, b-1,\, z)=z^{p+1}M(a,\, b,\, z)-(p+2-b)M_p(0;\, a,\, b,\, z), \tag{61}$$

and for the $U_p(0;\, a,\, b,\, z)$, from eqs (59), (60), and (61), we have

$$a(1+a-b)U_p(0;\, a+1,\, b,\, z)=z^{p+1}U(a,\, b,\, z)-(p+1-a)U_p(0;\, a,\, b,\, z), \tag{62}$$

$$(b-a-1)U_p(0;\, a,\, b-1,\, z)=z^{p+1}U(a,\, b,\, z)-(p+2-b)U_p(0;\, a,\, b,\, z). \tag{63}$$

Finally, by integrating eq (1) by parts, we have the recursion relations

$$aM_{p+1}(0;\, a+1,\, b+1,\, z)=b[z^{p+1}M(a,\, b,\, z)-(p+1)M_p(0;\, a,\, b,\, z)], \tag{64}$$

$$aU_{p+1}(0;\, a+1,\, b+1,\, z)=(p+1)U_p(0;\, a,\, b,\, z)-z^{p+1}U(a,\, b,\, z). \tag{65}$$

For $p=$ integer n, $n=0,\, 1,\, 2,\, \ldots$, $M_n(0;\, a,\, b,\, z)\equiv\Phi_n(a,\, b,\, z)$ and $U_n(0;\, a,\, b,\, z)\equiv\Psi_n(a,\, b,\, z)$, the cases discussed in sections 2 and 3 respectively. For $p\neq$ integer, four cases arise for which the integrals can be evaluated in terms of elementary functions or in terms of M's or U's respectively. The first case is the trivial one with $a=0$. We obviously have, by eqs (6) and (33)

$$M_p(0;\, 0,\, b,\, z)=U_p(0;\, 0,\, b,\, z)=\frac{z^{p+1}}{p+1}. \tag{66}$$

The second case, $a=p+2+m$, $m=0,\, 1,\, 2,\, \ldots$, is reduced by repeated use of eqs (60) and (62) to the lowest order terms $a=p+2$, where

$$M_p(0;\, p+2,\, b,\, z)=\frac{z^{p+1}}{p+1}M(p+1,\, b,\, z), \tag{67a}$$

$$U_p(0;\, p+2,\, b,\, z)=\frac{z^{p+1}}{(p+1)(p+2-b)}\,U(p+1,\, b,\, z), \tag{67b}$$

which are directly obtainable from eqs (60) and (62) by setting $a=p+1$. Thirdly, if $b=p+1-m$, then use of eqs (61) and (63) leads to the lowest order terms $b=p+1$, where

$$M_p(0;\, a,\, p+1,\, z)=\frac{z^{p+1}}{p+1}M(a,\, p+2,\, z), \tag{68a}$$

$$U_p(0;\, a,\, p+1,\, z)=\left(\frac{z^{p+1}}{p+1-a}\right)U(a,\, p+2,\, z), \tag{68b}$$

which we find from eqs (61) and (63) by setting $b=p+2$. Lastly, for $b-a=\pm$ integer, use of eqs (60) and (61) reduces M_p to the case $a=b$ for which

$$M_p(0;\, a,\, a,\, z)=\frac{z^{p+1}}{p+1}M(p+1,\, p+2,\, z), \tag{69a}$$

whereas, if $b-a=1,\, 2,\, \ldots$, repeated application of eqs (62) and (63) reduces U_p to the case $b=a+1$ for which

$$U_p(0;\, a,\, a+1,\, z)=\int z^{p-a}dz. \tag{69b}$$

We note that if none of the above conditions are satisfied, then use of eqs (60)–(65) will reduce the integrals to a series of similar integrals with p, a, b in the interval (0, 1). Further analytical reductions are not available.

5. The General Integrals $M_p(\alpha; a, b, z)$ and $U_p(\alpha; a, b, z)$

Let us consider the differential equation

$$z\frac{d^2y}{dz^2}+(b-z)\frac{dy}{dz}-ay=f(z). \tag{70}$$

For $f(z)=0$, we have the well-known confluent hypergeometric equation for which $M(a, b, z)$ and $U(a, b, z)$ are solutions. Babister [6] defines two nonhomogeneous confluent hypergeometric functions $\theta_\sigma(a, b; z)$ and $\Lambda_{\rho,\sigma}(a, b; z)$ which are particular solutions of eq (70) for (i) $f(z)=z^{\sigma-1}$ and (ii) $f(z)=\exp(z)z^{\sigma-1}$ respectively. He further discusses series expansions, contiguous relations, recurrence formulas, differential properties and several definite and indefinite integrals for θ and Λ.

We shall presently express M_p and U_p in terms of the functions investigated by Babister. Consider the following special cases of eq (70),

$$z\frac{d^2y}{dz^2}+(b-z)\frac{dy}{dz}-ay=\exp(\rho z)z^{\sigma-1}, \tag{70a}$$

and

$$z\frac{d^2u}{dz^2}+(b-z)\frac{du}{dz}-au=0, \tag{70b}$$

with particular solutions $y=\Lambda_{\rho,\sigma}(a, b; z)$ and $u=M(a, b, z)$, respectively. Multiplying eq (70a) by $ue^{-z}z^{b-1}$ and eq (70b) by $ye^{-z}z^{b-1}$, and subtracting, we obtain

$$\frac{d}{dz}\left[z^be^{-z}\left(M(a, b, z)\frac{d}{dz}\Lambda_{\rho,\sigma}(a, b; z)-\Lambda_{\rho,\sigma}(a, b; z)\frac{d}{dz}M(a, b, z)\right)\right]$$
$$=e^{(\rho-1)z}z^{b+\sigma-2}M(a, b, z). \tag{71}$$

Integrating both sides of eq (71) and setting $b+\sigma-2=p$ and $\rho=\alpha+1$, we have

$$M_p(\alpha; a, b, z)=z^be^{-z}\left[M(a, b, z)\frac{d}{dz}\Lambda_{\alpha+1,\,p-b+2}(a, b; z)\right.$$
$$\left.-\Lambda_{\alpha+1,\,p-b+2}(a, b; z)\frac{d}{dz}M(a, b, z)\right]. \tag{72}$$

From the differential properties of $M(a, b, z)$, eq (5), and $\Lambda_{\rho,\sigma}(a, b; z)$ [6, p. 135]

$$\frac{d}{dz}\Lambda_{\rho,\sigma}(a, b; z)=(\sigma-1)\Lambda_{\rho,\sigma-1}(a+1, b+1; z)+\rho\Lambda_{\rho,\sigma}(a+1, b+1; z), \tag{73}$$

we obtain an alternative expression for $M_p(\alpha; a, b, z)$ of eq (72)

$$M_p(\alpha; a, b, z)=z^be^{-z}\left[(\alpha+1)M(a, b, z)\Lambda_{\alpha+1,\,p-b+2}(a+1, b+1; z)\right.$$
$$+(p-b+1)M(a, b, z)\Lambda_{\alpha+1,\,p-b+2}(a+1, b+1; z)$$
$$\left.-\frac{a}{b}M(a+1, b+1, z)\Lambda_{\alpha+1,\,p-b+2}(a, b; z)\right] \tag{74}$$

Contiguous relations for the Λ are given by Babister [6, p. 135]. Two interesting special cases of eq (74) are obtainable, when $\alpha=0$ and when $\alpha=-1$ due to the relations [6, p. 134]

$$\Lambda_{0,\sigma}(a, b; z)=\theta_\sigma(a, b; z), \tag{75a}$$

$$\Lambda_{1,\sigma}(a, b; z)=e^{-i\pi\sigma}e^z\theta_\sigma(b-a, b, e^{i\pi}z). \tag{75b}$$

The first is obtained from eq (74) with $\alpha=0$ and eq (75b) and gives an expression for the $M_p(0;\ a,\ b,\ z)$ of section 4 in terms of θ's and M's

$$M_p(0;\ a,\ b,\ z)=z^b e^{i\pi(b-p)}\Big[M(a,\ b,\ z)\theta_{p-b+2}(b-a,\ b+1;\ e^{i\pi}z) - (p-b+1)M(a,\ b,\ z)\theta_{p-b+1}(b-a,\ b+1;\ e^{i\pi}z) - \frac{a}{b}M(a+1,\ b+1,\ z)\theta_{p-b+2}(b-a,\ b,\ e^{i\pi}z)\Big]. \tag{76}$$

Furthermore, from eq (57) and from the following expression [6, p. 121]

$$\theta_\sigma(a,\ b;\ z)=\frac{z^\sigma}{\sigma(\sigma+b-1)}\,{}_2F_2(1,\ \sigma+a;\ \sigma+1,\ \sigma+b;\ z), \tag{77}$$

we obtain

$${}_2F_2(p+1,\ a;\ p+2,\ b;\ z)=M(a,\ b,\ z)\,{}_2F_2(1,\ p-a+1;\ p-b+2,\ p+2;\ -z) + \frac{p+1}{p+2}\frac{z}{(p-b+2)}M(a,\ b,\ z)\,{}_2F_2(1,\ p-a+2;\ p-b+3,\ p+3;\ -z) - \frac{a}{b}\frac{z}{(p-b+2)}M(a+1,\ b+1,\ z)\,{}_2F_2(1,\ p-a+2;\ p-b+3,\ p+2;\ -z). \tag{78}$$

The second case of interest is obtained from eq (74) with $\alpha=-1$ and eq (75a), and is given by [6, p. 131]

$$M_p(-1;\ a,\ b,\ z)=z^b e^{-z}\Big[(p-b+1)M(a,\ b,\ z)\theta_{p-b+1}(a+1,\ b+1;\ z) - \frac{a}{b}M(a+1,\ b+1,\ z)\theta_{p-b+2}(a,\ b;\ z)\Big]. \tag{79}$$

$M_p(-1;\ a,\ b,\ z)$ can be related to the $M_p(0;\ a,\ b,\ z)$ discussed above and in section 4, if we use Kummer's relation [4, 5]

$$e^{-z}M(a,\ b,\ z)=M(b-a,\ b,\ e^{i\pi}z); \tag{80}$$

thus

$$M_p(-1;\ a,\ b,\ z)=e^{-i\pi(p+1)}M_p(0;\ b-a,\ b,\ e^{i\pi}z). \tag{81}$$

The condition $b=a$ in eq (81) gives

$$M_p(-1;\ a,\ a,\ z)=\frac{z^{p+1}}{p+1}, \tag{82a}$$

corresponding to the first case considered in section 4, eq (66). The integral of eq (80) in addition to eq (5b) gives

$$M_0(-1;\ a,\ b,\ z)=\frac{b-1}{a+1-b}e^{-z}M(a,\ b-1,\ z). \tag{82b}$$

Cases two and three of section 4, eqs (67a) and (68a), allow for the analytical integration of $M_p(-1; a, b, z)$ when $b=a+p+2+m$ and $b=p+1-m$ for $m=0, 1, 2, \ldots$, and reduce respectively to the lowest order terms

$$M_p(-1; a, a+p+2, z)=\frac{z^{p+1}}{p+1}e^{-z}M(a+1, a+p+2, z), \tag{82c}$$

$$M_p(-1; a, p+1, z)=\frac{z^{p+1}}{p+1}e^{-z}M(a+1, p+2, z). \tag{82d}$$

From case four of section 4 we can obtain the integrals $M_p(-1; \pm n, b, z)$ for $n=0, 1, 2, \ldots$. Specifically, for n positive we have

$$M_p(-1; n, b, z)=z^{p+1}e^{-z}\left[(-1)^n\frac{(b-n)_n}{(p-b+2)_n}\frac{1}{p+1}M(1, p+2, z)\right.$$
$$\left.+\sum_{k=0}^{n-1}(-1)^k\frac{(b-n)_k}{(p-b+n+1-k)_{k+1}}M(n-k, b, z)\right]. \tag{82e}$$

For n negative, rather than using the reduction formula eq (60), it is simpler to integrate term by term the finite polynomial $M(-n, b, z)$.

The analysis proceeds in a straight forward manner for the $U_p(\alpha; a, b, z)$ if we use the definition of $U(a, b, z)$ in terms of M from eq (29). We then have

$$U_p(\alpha; a, b, z)=\frac{\pi}{\sin \pi b}\left[\frac{1}{\Gamma(1+a-b)\Gamma(b)}M_p(\alpha; a, b, z)\right.$$
$$\left.-\frac{1}{\Gamma(a)\Gamma(2-b)}M_{p-b+1}(\alpha; a-b+1, 2-b, z)\right]. \tag{83}$$

We first note that for $b=a+1$

$$U_p(\alpha; a, a+1, z)=M_{p-a}(\alpha; 0, 1-a, z)=\int z^{p-a}e^{\alpha z}dz. \tag{84a}$$

We also obtain from eq (83) for $\alpha=-1$ and eqs (26) and (82e) the case $b=a$

$$U_p(-1; a, a, z)=\frac{1}{p+1}[z^{p+1}\Gamma(1-a, z)+\gamma(p-a+2, z)]. \tag{84b}$$

Analogous to eq (82b) we find that by repeated use of eq (31), $U_n(-1; a', b', z)$ is reduced to a series of terms $U_0(-1; a, b, z)$, given by

$$U_0(-1; a, b, z)=-e^{-z}U(a, b-1, z). \tag{84c}$$

Corresponding to the cases $b=a+p+2+m$ and $b=p+1-m$ we have

$$U_p(-1; a, a+p+2+m, z)=-z^{p+1}e^{-z}\sum_{k=0}^{m}(-1)^k\frac{m!}{(m-k)!}\ U(a+1+k, a+p+2+m, z), \tag{84d}$$

$$U_p(-1; a, p+1-m, z)=-z^{p+1}e^{-z}\sum_{k=0}^{m}\frac{m!}{(m-k)!}\ U(a+1+k, p+2-m+k, z). \tag{84e}$$

We note that for $b=\pm n$, care must be taken in the use of eqs (59) and (83). In this instance, the limiting forms for these equations must be used. For a detailed discussion of this point, the reader is referred to Erdélyi et al. [9].

6. Appendix: Reduction to the Bessel Function Case

If $b=2a$, then $M(a, b, z)$ and $U(a, b, z)$ are related to the modified Bessel functions I and K

$$M\left(\nu+\frac{1}{2}, 2\nu+1, 2z\right)=\Gamma(\nu+1)2^{\nu}z^{-\nu}e^{z}I_{\nu}(z), \tag{A1}$$

$$U\left(\nu+\frac{1}{2}, 2\nu+1, 2z\right)=\pi^{-1/2}(2z)^{-\nu}e^{z}K_{\nu}(z). \tag{A2}$$

Specializing the parameters in eq (57) to the case $b=2a$ and substituting from eq (A1), we have the indefinite integral

$$\int z^{\mu}e^{\pm z}I_{\nu}(z)dz=\frac{z^{\mu+\nu+1}}{(\mu+\nu+1)\Gamma(\nu+1)}2^{-\nu}{}_2F_2\left(\mu+\nu+1, \nu+\frac{1}{2}; \mu+\nu+2, 2\nu+1; \pm 2z\right), \tag{A3}$$

in agreement with Luke [1, p. 96]. Equations (67a) and (82c) give special cases of eq (A3) with $\mu=-3/2$ and $\nu=p+(3/2)$ whereas eqs (68a) and (82d) give special cases of eq (A3) with $\mu=\nu=p/2$.

Equations (67b) and (84d) of the text give rise to

$$\int z^{-3/2}e^{z}K_{p+(3/2)}(z)dz=-\sqrt{2\pi}\,\frac{(2z)^{p+1}}{(p+1)(p+2)}U(p+1, 2p+4, 2z)$$

$$=\frac{z^{-1/2}e^{z}}{(p+1)(p+2)}\left[(p+2-z)K_{p+(3/2)}(z)-zK_{p+(5/2)}(z)\right], \tag{A4}$$

$$\int z^{-3/2}e^{-z}K_{p+(3/2)}(z)dz=-\sqrt{2\pi}\,(2z)^{p+1}e^{-2z}U(p+3, 2p+4, 2z)$$

$$=\frac{z^{-1/2}e^{-z}}{(p+1)(p+2)}\left[(p+2+z)K_{p+(3/2)}(z)-zK_{p+(5/2)}(z)\right]; \tag{A5}$$

whereas eqs (68b) and (84e) yield

$$\int z^{\nu}e^{z}K_{\nu}(z)dz=\sqrt{\pi}\,\frac{(2z)^{\nu+1}}{(2\nu+1)}z^{\nu}U(\nu+1/2, 2\nu+2, 2z)$$

$$=\frac{e^{z}z^{\nu+1}}{(2\nu+1)}\left[K_{\nu}(z)+K_{\nu+1}(z)\right], \tag{A6}$$

$$\int z^{\nu}e^{-z}K_{\nu}(z)dz=-\sqrt{\pi}\,(2z)^{\nu}z^{\nu+1}e^{-2z}U(\nu+3/2, 2\nu+2, 2z)$$

$$=\frac{e^{-z}z^{\nu+1}}{(2\nu+1)}\left[K_{\nu}(z)-K_{\nu+1}(z)\right], \tag{A7}$$

in agreement with the Handbook of Mathematical Functions.[3] We note further that eq (84b) gives

$$\int e^{-z}z^{-\nu}K_{\nu}(z)dz=\frac{1}{(2\nu-1)}e^{-z}z^{1-\nu}\left[K_{\nu-1}(z)-K_{\nu}(z)\right] \qquad (\nu\neq 1/2). \tag{A8}$$

From eqs (74) and (78) we obtain indefinite integrals of the product of powers of z, Bessel functions, and $\exp(\alpha z)$ expressed in terms of the Λ's and θ's of section 5. Equation (79) leads to

[3] See eqs. (11.3.15) and (11.3.16) in [4, chap. 11]. Luke [1, p. 97] gives an equivalent form for (A6) and (A7) when $\nu=0$.

an interesting identity involving ${}_2F_2$'s

$$\left(\frac{z}{2}\right)^{\nu} e^{z}[\Gamma(\nu+1)]^{-1}\,{}_2F_2(\nu+1/2,\ \mu+\nu+1;\ 2\nu+1,\ \mu+\nu+2;\ -2z)$$
$$= I_{\nu}(z)\,{}_2F_2\left(1,\ \mu+\frac{3}{2};\ \mu-\nu+1,\ \mu+\nu+2;\ 2z\right)$$
$$-\frac{z}{(\mu-\nu+1)}\left[I_{\nu}(z)+I_{\nu+1}(z)\right]{}_2F_2\,(1,\ \mu+\frac{3}{2};\ \mu-\nu+2,\ \mu+\nu+2;\ 2z). \qquad \text{(A9)}$$

Finally, from eq (82b), we obtain a decomposition formula for ${}_2F_2\ (a,\ b;\ c,\ d;\ z)$ when $a=1$, $c=2$, and $d=2b$. We then have [4]

$$z\,{}_2F_2(1,\ a+1;\ 2,\ 2a+2;\ z)=\frac{2}{a}\,(a+1/2)\,[M(a,\ 2a+1,\ z)-1], \qquad \text{(A10a)}$$

for $a \neq -k/2$, $k=0, 1, 2, \ldots$, or in terms of Bessel functions

$$2z\,{}_2F_2(1,\ \nu+1/2;\ 2,\ 2\nu+1;\ 2z)=\frac{1}{\left(\nu-\frac{1}{2}\right)}\,[(2/z)^{\nu} z e^{z}\Gamma(\nu+1)\{I_{\nu-1}(z)-I_{\nu}(z)\}-2\nu]. \qquad \text{(A10b)}$$

7. References

[1] Luke, Y. L., Integrals of Bessel Functions (McGraw-Hill Book Co., New York, 1962).

[2] Kim, H., and Parr, R. G., One-Center Perturbation Approach to Molecular Electronic Energies, J. Chem. Phys. **49**, 3071 (1968).

[3] Pan, Y. C., and Hameka, H. F., Use of Green Functions in Atomic and Molecular Calculations, J. Chem. Phys. **49**, 2009 (1968).

[4] Handbook of Mathematical Functions, edited by M. Abramowitz and I. A. Stegun, National Bureau of Standards, Applied Mathematics Series No. 55, U.S. Government Printing Office, Washington, D.C., June (1964). See Chapter 13, pages 503–536, on Confluent Hypergeometric Functions.

[5] Slater, L. J., Confluent Hypergeometric Functions (Cambridge University Press, London, 1960).

[6] Babister, A. W., Transcendental Functions (The Macmillan Co., New York, 1967).

[7] Erdélyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G., Higher Transcendental Functions (McGraw-Hill Book Co., New York, 1953), Vol. **2**. See Chapter 9 for an extensive discussion of the incomplete gamma functions.

[8] Rainville, E. D., Special Functions (The Macmillan Co., New York (1960), pp. 82–84.

[9] Erdélyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G., Higher Transcendental Functions (McGraw-Hill Book Co., New York, 1953), Vol. **1**. See Chapter 6, section 6.7.1 for a discussion of the logarithmic case.

(Paper 74B2–321)

[4] Equation (A10a) is a particular example of the more general result derivable from the series representation of ${}_pF_q$.

$$z_{p+1}F_{q+1}(1,\ a_p+1;\ 2,\ b_q+1;\ z)=[{}_pF_q(a_p;\ b_q;\ z)-1]\left(\prod_{i=1}^{p} a_i\right)\Big/\left(\prod_{i=1}^{q} b_i\right).$$

JOURNAL OF RESEARCH of the National Bureau of Standards – B. Mathematical Sciences
Vol. 74B, No. 2, April–June 1970

On Complementary Polar Conical Sets*

Christoph Witzgall**

(February 6, 1970)

Tucker has formulated the Duality Theorem of Linear Programming in terms of orthogonality properties of a pair of complementary orthogonal linear manifolds with respect to the positive orthant. This theorem is generalized by substituting complementary polar conical sets for complementary orthogonal linear manifolds, and the generalization is proved under simple stability assumptions. Equivalence to Fenchel's Duality Theorem for conjugate convex functions is established. There are strong parallelisms to work by Kretschmer.

Key words: Conjugate functions; convex cones; duality; linear programming; orthogonality; polarity.

1. Introduction

Two linear manifolds M and N in R^n are called *orthogonal*, if $(X_1-X_2)^T(Y_1-Y_2)=0$ holds whenever $X_1, X_2 \epsilon M$ and $Y_1, Y_2 \epsilon N$. They are called *complementary orthogonal*, if they are orthogonal, if their intersection is of dimension zero, and if their dimensions add up to n.

The following formulation of the duality theorem of Linear Programming has been given by Tucker [1]:[1]

THEOREM M: *Suppose* M *and* N *are complementary orthogonal linear manifolds both of which meet the nonnegative orthant* $R^n_{\oplus}$. *Then there exist two nonnegative vectors* U *and* V *such that* $U \epsilon M$, $V \epsilon N$, *and* $U^T V = 0$ (fig. 1).

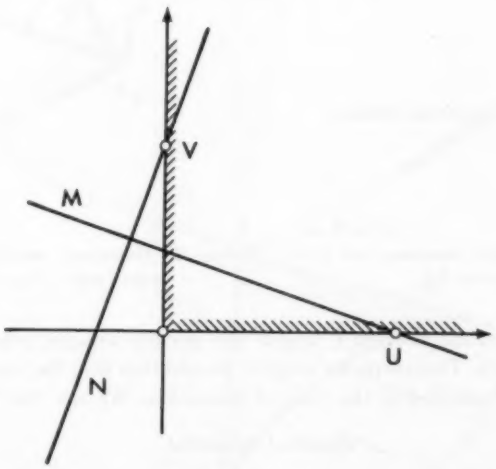

FIGURE 1. *Orthogonal solutions* U, V *for complementary orthogonal manifolds* M, N.

*An invited paper.

**Present address: Boeing Scientific Research Laboratories, Mathematics Research Laboratory, Seattle, Washington 98111.

[1] Figures in brackets indicate the literature references at the end of this paper.

382-233 O - 70 - 3

The purpose of this paper is to extend the above theorem to pairs of "complementary polar conical sets."

We say that two sets F and G are

complementary polar conical sets,

if $F=P+C$ and $G=Q+C^p$ for some points P, Q, and some cone C with $C=C^{pp}$. Here C^p denotes the (negative) *polar* of the cone C, i.e.,

$$C^p=\{Y \mid Y^TX \leq 0 \text{ for all } X \epsilon C\}.$$

We want to consider the existence of nonnegative vectors U and V such that $U \epsilon F$, $V \epsilon G$, and $U^TV=0$, where F and G are complementary polar conical sets. This statement generalizes *Theorem M*, since each pair of complementary orthogonal manifolds is also a pair of complementary polar conical sets (fig. 2). Carrying the generalization of *Theorem M* still further, we drop the nonnegativity hypothesis for U and V and require instead that

$$U \epsilon K,\ V \epsilon -K^p$$

for some given (closed) cone K (fig. 3).

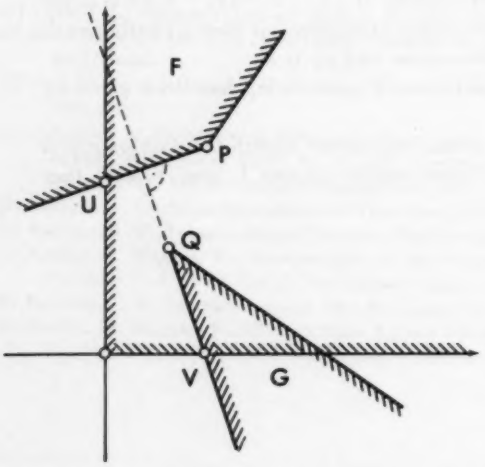

FIGURE 2. *Orthogonal solutions* U, V *for complementary polar conical sets* F, G *with respect to* $R^n_\oplus$.

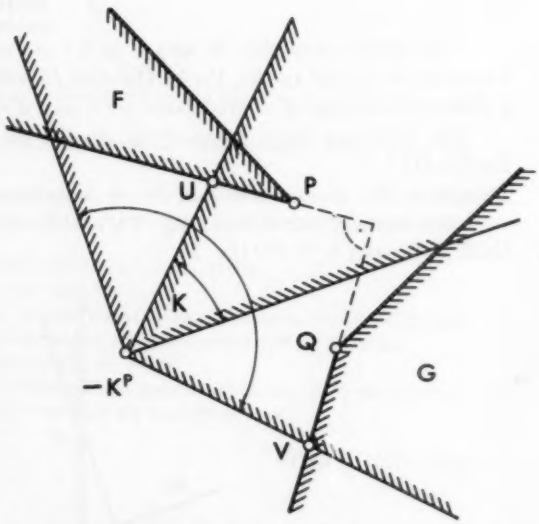

FIGURE 3. *Orthogonal solutions* U, V *for complementary polar conical sets* F, G *with respect to cones* K, $-K^p$.

In the case of manifolds, the vectors U and V are usually unique, whereas they are typically *not* unique in all other cases. Therefore we require in addition that the vectors $U-P$ and $V-Q$ be orthogonal—a condition satisfied in the case of manifolds. We say that U and V form a pair of

orthogonal solutions

of the complementary polar sets F and G with respect to the cone K.

It would appear natural to generalize even further by adding points P', Q' to K and $-K^p$, respectively. Orthogonal solutions would then be defined as satisfying

$$U \epsilon (P+C) \cap (P'+K),\ V \epsilon (Q+C^p) \cap (Q'-K^p)$$

$$(U-P)^T(V-Q)=0=(U-P')^T(V-Q').$$

However, the linear transformations $U:=U-P'$, $V:=V-Q'$, $\bar{P}:=P-P'$, $\bar{Q}:=Q-Q'$ reduce this case immediately to

$$\bar{U}\epsilon(\bar{P}+C)\cap K,\ \bar{V}\epsilon(\bar{Q}+C^p)\cap(-K^p),$$

$$(\bar{U}-\bar{P})^T(\bar{V}-\bar{Q})=0=\bar{U}^T\bar{V}.$$

Therefore it is no restriction of generality to assume $P'=Q'=0$, and this will be done throughout this paper.

2. Propositions

The first extension of *Theorem M* concerns the special class of *polyhedral* conical sets, that is, conical sets that are intersections of finitely many closed halfspaces. If $F=P+C$ is such a conical set, then so is $G=Q+C^p$, and $C=C^{pp}$. Here R may be any ordered field.

THEOREM P: *If* F *and* G *are complementary polar polyhedral conical sets both of which meet polyhedral cones* K *and* $-$K^p *respectively, then* F, G *possess orthogonal solutions* UϵK, V$\epsilon-$K^p.

If R is the field of real numbers, then the relation $C=C^{pp}$ characterizes closed cones. The following example shows that general complementary polar conical sets need not admit orthogonal solutions, even if both meet the nonnegative orthant $K=-K^p=R^n_+$. Consider

$$C=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\mid -y+z\geqq\sqrt{x^2+y^2+z^2}\right\}.$$

This is a circular cone with an opening angle of 90°. Clearly $C^p=-C$. Choose $P=Q=(1,0,0)^T$. Then (fig. 4)

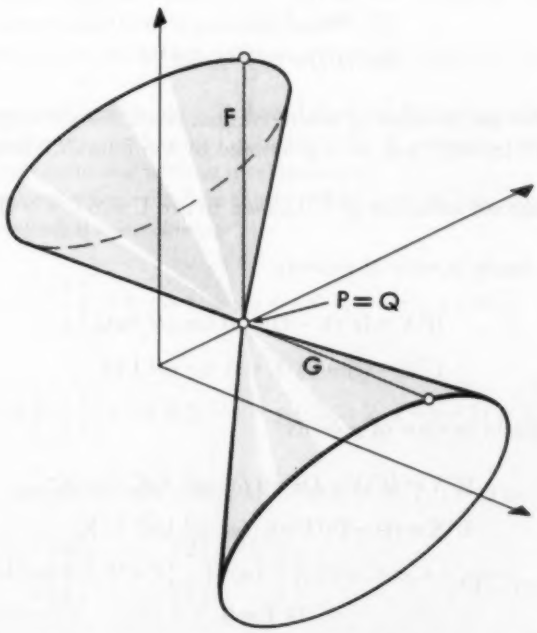

FIGURE 4. *Example for nonexistence of orthogonal solutions: Two circular cones with angle of 90° at apex, meeting the positive orthant in two rays* F *and* G.

$$F \cap R_{\oplus}^{n} = \{P + \theta(0, 0, 1)^{T} \mid \theta \geq 0\}$$
$$G \cap R_{\oplus}^{n} = \{Q + \theta(0, 1, 0)^{T} \mid \theta \geq 0\}$$

and these sets have no orthogonal points in common.

Thus we need additional conditions in order to extend *Theorem M* to general conical sets in the real space R^n.

A point D of a closed cone C is called

stable in C

if the conical hull $\mathscr{C}(C \cup (-D))$ is again closed.

If D lies in the relative interior of C, which we shall denote by C^I, then D is stable in C. Indeed,

$$D = \sum_{i=1}^{k} A_i u_i$$

where $A_i \epsilon C$, $u_i > 0$, and $k = \dim \mathscr{L}(C)$, where $\mathscr{L}(C)$ denotes the linear hull of C. Thus $A_1, \ldots, A_k, -D$ span positively the linear hull of C. If therefore $D \epsilon C^I$ then $\mathscr{C}(C \cup (-D)) = \mathscr{L}(C)$, and linear subspaces of R^n are closed.

If D belongs to a polyhedral cone C, then $\mathscr{C}(C \cup (-D))$ is again a polyhedral cone and therefore closed. Thus every point of a polyhedral cone C is stable in C.

Further we need the notion of a

direction of infinity.

Vector D is a direction of infinity of F with respect to Q, and vector E a direction of infinity of G with respect to P, F

$$\begin{aligned} &\text{(i)} \quad D \epsilon K, \; E \epsilon -K^{p} \\ &\text{(ii)} \quad D \epsilon C, \; E \epsilon C^{p} \\ &\text{(iii)} \quad D^{T}Q = 0, \; P^{T}E = 0, \end{aligned} \tag{1}$$

respectively. Zero vectors are included in this definition, since this will simplify statements later on. The name "directions of infinity" has been prompted by the following fact:

If U, V *are orthogonal solutions of* F, G, *then so are* $U + \theta D$, $V + \tau E$ *for all* $\theta, \tau \geq 0$. (2)

Indeed, (1.ii) and (1.iii) imply in view of polarity:

$$\begin{aligned} &D^{T}V = D^{T}(V - Q) \leq 0 \; \textit{for all} \; V \epsilon G \\ &U^{T}E = (U - P)^{T}E \leq 0 \; \textit{for all} \; U \epsilon F. \end{aligned} \tag{3}$$

From (1.i) one obtains again in view of polarity:

$$\begin{aligned} &D^{T}V = D^{T}(V - Q) = 0 \; \textit{for all} \; V \epsilon G \cap -K^{p} \\ &U^{T}E = (U - P)^{T}E = 0 \; \textit{for all} \; U \epsilon F \cap K. \end{aligned} \tag{4}$$

Finally, since $E + Q \epsilon G \cap -K^{p}$,

$$D^{T}E = 0. \tag{5}$$

Proposition (2) now follows from (4) and (5): $(U + \theta D)^{T}(V + \tau E) = U^{T}V = 0$, $(U + \theta D - P)^{T}(V + \tau E - Q) = (U - P)^{T}(V - Q) = 0$. $U + \theta D \epsilon F \cap K$ and $V + \tau E \epsilon G \cap -K^{p}$ is plain, as $D \epsilon C \cap K$ and $E \epsilon C^{p} \cap -K^{p}$.

We are now able to formulate the general

THEOREM S: Suppose $F=P+C$ *and* $G=Q+C^p$ *are a pair of complementary polar sets. If* F *and* G *both meet the closed cones* K *and* $-K^p$ *respectively, and if all* [2] *directions of infinity of* F *are stable in* C *and* K, *while all directions of infinity of* G *are stable in* C^p *and* $-K^p$, *then* F, G *possess orthogonal solutions* $U\epsilon K$, $V\epsilon -K^p$.

Let $\mathcal{M}(S)$ denote the affine hull of a set $S\subseteq R^n$ and let S^I stand for the interior of S with respect to the relative topology in $\mathcal{M}(S)$. We call S^I the *relative interior* of S. Then we have as a corollary to *Theorem S*:

THEOREM I: *If* F *and* G *are complementary polar conical sets in the real space* R^n, *and if* $F^I\cap K^I\neq\emptyset$, $G^I\cap(-K^p)^I\neq\emptyset$, *then* F *and* G *possess orthogonal solutions* $U\epsilon K$, $V\epsilon -K^p$.

This follows immediately from the following

LEMMA: *If* $G^I\cap(-K^p)^I\neq\emptyset$, *then* $-D\epsilon C\cap K$ *holds for every direction of infinity* D *of* $F=P+C$.

PROOF: By (3), the plane $H=\{X\mid D^TX=0\}$ is a supporting plane of G. Since $G^I\cap(-K^p)^I\neq\emptyset$ by hypothesis and $H\supseteq G\cap(-K^p)$ by (4), we have $H\cap G^I\neq\emptyset$ and $H\cap(-K^p)^I\neq\emptyset$. Any supporting plane of a convex set S which meets the relative interior S^I must contain S. Hence $H\supseteq G$ and $H\supseteq -K^p$. In view of (1.iii) we have $H\supseteq G-Q=C^p$. This is equivalent to $-D\epsilon C$. $-D\epsilon K$ follows from $H=-H\supseteq K^p$. —[3]

Theorem P for the real space R^n also follows from *Theorem S*. Indeed, if C and K are polyhedral cones, they have a finite number of generators, as have C^p and $-K^p$. But then so have $\mathcal{C}(C\cup\{-D\})$, $\mathcal{C}(K\cup\{-D\})$, $\mathcal{C}(C^p\cup\{-E\})$, $\mathcal{C}(-K^p\cup\{-E\})$, which are therefore again polyhedral cones and therefore closed, no matter what D and E are selected.

3. Uniqueness

Let us briefly examine the uniqueness of orthogonal solutions U, V, however only in the case $K=-K^p=R^n_\oplus$. If there exists an index i such that $u_i=v_i=0$, we say that U and V have a *common zero*. In the case of manifolds, there is a known result:

THEOREM U: *Orthogonal solutions with a common zero exist if and only if the orthogonal solutions are not unique.*

The "only if" part of *Theorem U* is commonly deduced from the following statement:

THEOREM V: *If complementary orthogonal manifolds* M, N *possess orthogonal solutions at all, then they possess orthogonal solutions without a common zero.*

The analogous statements for complementary polar conical sets do not hold. Consider for example the following polyhedral conical sets:

$$F:=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\;\middle|\; x+y\geq 3,\ x-y+z\leq 1,\ -x+y+z\leq 1\right\}$$

$$G:=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\;\middle|\; x+y\leq 3,\ -x+y+2z\geq 2,\ x-y+2z\geq 2\right\}.$$

The above two sets are complementary polar, with $P=Q=(\frac{3}{2},\frac{3}{2},1)^T$. The orthogonal solutions are

$$V=\begin{pmatrix}0\\0\\1\end{pmatrix},\ U=\lambda\begin{pmatrix}2\\1\\0\end{pmatrix}+\mu\begin{pmatrix}1\\2\\0\end{pmatrix},\ \lambda,\ \mu\geq 0,\ \lambda+\mu=1.$$

[2] It will become apparent from the proof of Theorem S, that it suffices to require that there exists at least one stable direction in the relative interior of the cone of directions of infinity.

[3] The horizontal bar at the end of a paragraph marks termination of a proof.

They are not unique in spite of the absence of common zeros. Thus the "if" part of *Theorem U* is not valid for complementary polar conical sets.

As a counterexample for *Theorem V* consider the conical sets:

$$F=C:=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\,\middle|\,-y+z\geq\sqrt{x^2+y^2+z^2}\right\},\ G=C^p:=-C.\ K=-K^p=R^3_{\oplus}.$$

Here $F\cap R^3_{\oplus}$ and $G\cap R^3_{\oplus}$ are the nonnegative parts of the z axis and the y axis, respectively. Hence $x=0$ for all $U\epsilon F\cap R^3_{\oplus}$ and $V\epsilon G\cap R^3_{\oplus}$. Note that in this example the existence of orthogonal solutions cannot be inferred from *Theorem S*.

4. Proofs

We proceed to prove *Theorem P*. The proof of *Theorem S* will then be based on *Theorem P*.

PROOF OF THEOREM P: Let $U\epsilon K$, $V\epsilon -K^p$ be vectors in F, G respectively. By the definition of the polar $(U-P)^T(V-Q)\leq 0$ as well as $U^TV\geq 0$. Thus

$$U^TV-(U-P)^T(V-Q)=U^TQ+P^TV-P^TQ\geq 0 \tag{6}$$

for $U\epsilon F\cap K$ and $V\epsilon G\cap(-K^p)$, and

$$U^TQ+P^TV-P^TQ=0$$

is necessary and sufficient for $U^TV=0$ and $(U-P)^T(V-Q)=0$ to hold simultaneously. The existence of a pair of orthogonal solutions U and V is therefore equivalent to the following linear program having an optimal value of zero:

$$\begin{aligned}&\textit{Minimize}\ \mathrm{U^TQ+P^TV-P^TQ}\ \textit{over}\ \mathrm{U,\ V}\ \textit{subject to}\\&\mathrm{U\epsilon P+C,\ U\epsilon K}\\&\mathrm{V\epsilon Q+C^p,\ V\epsilon -K^p.}\end{aligned}\tag{7}$$

Program (7) is *separable:* The pair U, V is an optimal solution of (7) if and only if U and V are optimal solutions of the following two programs respectively:

$$\begin{aligned}&\textit{Minimize}\ \mathrm{U^TQ}\ \textit{over}\ \mathrm{U}\ \textit{subject to}\\&\mathrm{U\epsilon P+C,\ U\epsilon K,}\end{aligned}\tag{8}$$

and

$$\begin{aligned}&\textit{Minimize}\ \mathrm{P^TV-P^TQ}\ \textit{over}\ \mathrm{V}\ \textit{subject to}\\&\mathrm{V\epsilon Q+C^p,\ V\epsilon -K^p.}\end{aligned}\tag{9}$$

By hypothesis, C and K are polyhedral cones

$$C=\{S\,|\,A^TS\leq 0\},\ K=\{BZ\,|\,Z\geq 0\}$$

$$C^p=\{AT\,|\,T\geq 0\},\ -K^p=\{W\,|\,B^TW\geq 0\}.$$

One has $U=BZ=P+S$ with $Z\geq 0$, $A^TS\leq 0$, and $V=Q+AT$ with $T\geq 0$, $B^TV\geq 0$. In terms of

Z and T the programs (8) and (9) thus reduce to

$$\begin{gathered} \textit{Minimize } \mathrm{Z^T B^T Q} \textit{ with } \mathrm{Z} \textit{ subject to} \\ -\mathrm{A^T BZ} \geqslant -\mathrm{A^T P} \\ \mathrm{Z} \geqslant 0. \end{gathered} \tag{10}$$

$$\begin{gathered} \textit{Maximize } -\mathrm{P^T AT} \textit{ with } \mathrm{T} \textit{ subject to} \\ -\mathrm{B^T AT} \leqslant \mathrm{B^T Q} \\ \mathrm{T} \geqslant 0. \end{gathered} \tag{11}$$

The programs (10) and (11) are clearly *duals* of each other. The duality theorem of linear programming then gives $Z^TB^TQ=-P^TAT$ or

$$U^TQ+P^TV-P^TQ=0,$$

as $U^T=Z^TB^T$ and $AT=V-Q$, for optimal solutions Z and T of programs (10) and (11), respectively. The existence of optimal solutions follows from the hypothesis, that F and G meet K and $-K^p$, respectively, and that therefore feasible solutions to both programs exist.–

PROOF OF THEOREM S: Let D_F be the set of all directions of infinity of $F=P+C$, and similarly let D_G be the set of all directions of infinity of $G=Q+C^p$. Plainly, D_F and D_G are closed convex cones. Moreover, by the stability hypothesis of *Theorem S*,

the four cones (12)

$$\begin{aligned} &\mathscr{C}(C \cup -D_F)=C-D_F=C+L_F,\ L_F:=\mathscr{L}(D_F)=D_F-D_F,\\ &\mathscr{C}(C^p \cup -D_G)=C^p-D_G=C^p+L_G,\ L_G:=\mathscr{L}(D_G)=D_G-D_G,\\ &\mathscr{C}(K \cup -D_F)=K-D_F=K+L_F,\\ &\mathscr{C}(-K^p \cup -D_G)=-K^p-D_G=-K^p+L_G \end{aligned}$$

are closed.

To see, for instance that $C+L_F$ is closed, let D_0 be any point in the relative interior D_F^I of D_F. Then $\mathscr{C}(D_F \cup \{-D_0\})=L_F$ and therefore $C+L_F=\mathscr{C}(C \cup \{-D_0\})$. The latter cone is closed since D_0 is stable by hypothesis.–

Next we show that without loss of generality, we may assume that

$$C \cap K \supseteq L_F \textit{ and } C^p \cap (-K^p) \supseteq L_G. \tag{13}$$

To prove this we shall use on several occasions the following simple lemma whose verification is left to the reader:

Let L *be a linear manifold in* $\mathrm{R^n}$, *and let* A *and* B *be any sets in* $\mathrm{R^n}$. *Then* $\mathrm{A} \subseteq \mathrm{L}$ *implies*

$$\mathrm{A}+(\mathrm{B} \cap \mathrm{L})=(\mathrm{A}+\mathrm{B}) \cap \mathrm{L}. \tag{14}$$

We return to proving that (13) constitutes no restriction of generality. Suppose the pairs of cones C, C^p and $K, -K^p$ do not satisfy (13). Then we consider the following modified pairs

$$\begin{aligned} &\hat{C}:=(C+L_F) \cap L_G^{\perp},\ \hat{C}^p:=(C^p+L_G) \cap L_F^{\perp}.\\ &\hat{K}:=(K+L_F) \cap L_G^{\perp},\ -\hat{K}^p:=(-K^p+L_G) \cap L_F^{\perp}. \end{aligned}$$

The (closed) cones $\hat{C}$ and $\hat{C}^p$ are indeed polars of each other, as are $\hat{K}$ and $\hat{K}^p$, since by (5), we have

$$L_F \subseteq L_G^{\perp} \textit{ and } L_G \subseteq L_F^{\perp}, \tag{15}$$

and (14) then gives

$$\hat{C} = (C \cap L_G^{\perp}) + L_F,$$

$$\hat{K} = (K \cap L_G^{\perp}) + L_F.$$

Polarizing these equations yields in view of (12)

$$\hat{C}^p = \overline{(C^p + L_G)} \cap L_F^{\perp} = (C^p + L_G) \cap L_F^{\perp}$$

$$\hat{K}^p = \overline{(K^p + L_G)} \cap L_F^{\perp} = (K^p + L_G) \cap L_F^{\perp},$$

and finally

$$-\hat{K}^p = (-K^p - L_G) \cap -(L_F^{\perp}) = (-K^p + L_G) \cap L_F^{\perp}.$$

Next we verify that $\hat{C}$, $\hat{C}^p$ and $\hat{K}$, $-\hat{K}^p$ indeed satisfy (13). To this end we denote by $\hat{C}_F$, $\hat{C}_G$ the cones of directions of infinity of $\hat{F} := P + \hat{C}$, $\hat{G} := Q + \hat{C}^p$ with respect to $\hat{K}$, $-\hat{K}^p$. Let then

$\hat{D} \in \hat{C}_F \subseteq \hat{C} \cap \hat{K}$. As $C + L_F = C - C_F$, since $L_F = C_F - C_F$ and $C_F \subseteq C$, we have $\hat{D} = D_1 - D_2$ where $D_1 \in C$ and $D_2 \in C_F$. Similarly, $\hat{D} = D_3 - D_4$ where $D_3 \in K$ and $D_4 \in C_F$. We claim $D_1 + D_4 \in C_F$. Indeed, $(D_1 + D_4)^T Q = (\hat{D} + D_2 + D_4)^T Q = \hat{D}^T Q + D_2^T Q + D_4^T Q = 0$; $D_1 + D_4 \in C$ as $D_1 \in C$, $D_4 \in C_F \subseteq C$; $D_1 + D_4 = D_2 + D_3 \in K$ as $D_2 \in C_F \subseteq K$, $D_3 \in K$. Clearly, $D_2 + D_4 \in C_F$. Thus $\hat{D} = (D_1 + D_4) - (D_2 + D_4) \in C_F - C_F = L_F$. This proves $\hat{C}_F \subseteq L_F$. Now $L_F \subseteq \hat{C} \cap \hat{K}$ by the definitions of $\hat{C}$ and $\hat{K}$, and in view of $L_F \subseteq L_G^{\perp}$ (15). Moreover, $\hat{D}^T Q = 0$ for every $\hat{D} \in L_F = C_F - C_F$ by (1.iii). Thus, and by an analogous argument for $\hat{G}$,

$$\hat{C}_F = L_F \text{ and } \hat{C}_G = L_G. \tag{16}$$

This clearly implies (13) for $\hat{C}$ and $\hat{K}$. Moreover it shows that $\mathscr{C}(\hat{C} \cup \{-\hat{D}\}) = \hat{C}$, $\mathscr{C}(\hat{C}^p \cup \{-\hat{E}\}) = \hat{C}^p$ for all $\hat{D} \in \hat{C}_F$ and $\hat{E} \in \hat{C}_G$. In other words, all directions in infinity of $\hat{F}$, $\hat{G}$ are stable (the reader should recall at this point, that the construction of F, G did require that the original sets F, G had only stable directions of infinity).

By (4),

$$F \cap K \subseteq L_G^{\perp} \text{ and } G \cap (-K^p) \subseteq L_F^{\perp}. \tag{17}$$

Since $P \in L_G$ by (1.iii), application of (14) yields

$$\hat{F} = P + \hat{C} = P + (C + L_F) \cap L_G^{\perp} = (P + C + L_F) \cap L_G^{\perp}.$$

Hence,

$$\hat{F} \cap \hat{K} = (P + C + L_F) \cap (K + L_F) \cap L_G^{\perp} \supseteq F \cap K \cap L_G^{\perp}$$

By (17), and an analogous argument for G,

$$\hat{F} \cap \hat{K} \supseteq F \cap K \text{ and } \hat{G} \cap (-\hat{K}^p) \supseteq G \cap (-K^p). \tag{18}$$

This shows that if F, G meet K, $-K^p$, respectively, then $\hat{F}$, $\hat{G}$ meet $\hat{K}$, $-\hat{K}^p$.

Altogether, we have seen that if F, G, K satisfy the hypotheses of *Theorem S*, then so do $\hat{F}$, $\hat{G}$, $\hat{K}$. It remains to be shown that if $\hat{F}$, $\hat{G}$ possess orthogonal solutions $\hat{U} \in \hat{K}$, $\hat{V} \in -\hat{K}^p$, then orthog-

onal solutions $U\epsilon K$, $V\epsilon -K^p$ of F, G can be derived from them. As $\hat{U}\epsilon\hat{F}=P+C-C_F=F-C_F$, *there exist* $U_1\epsilon F$, $D_1\epsilon C_F$ such that $\hat{U}=U_1-D_1$. As $\hat{U}\epsilon\hat{K}$, there exist $U_2\epsilon K$ and $D_2\epsilon C_F$ such that $\hat{U}=U_2-D_2$. Then $U:=U_1+D_2=U_2+D_1$ belongs to $F\cap K$. Similarly, there exist $V_1\epsilon G$, $V_2\epsilon -K^p$ and E_1, $E_2\epsilon C_G$ such that $V:=V_1+E_2=V_2+E_1\epsilon G\cap(-K^p)$. U and V satisfy the orthogonality conditions in view of (4) and (5). Hence they are orthogonal solutions of F, G with respect to K. If therefore *Theorem S* holds for cones satisfying (13), then it holds in general.

Consider now a pair of complementary polar conical sets $F=P+C$ and $G=Q+C^p$ with the cones C and K satisfying (13). Moreover, we suppose that $F\cap K\neq\emptyset$ and $G\cap -K^p\neq\emptyset$. Then we approximate C and K by increasing sequences of polyhedral cones $C^{(k)}$ and $K^{(k)}$,

$$\begin{aligned} C^{(1)}\subseteq C^{(2)}\subseteq\ \ldots\ \subseteq C=\overline{\cup\, C^{(k)}}\\ K^{(1)}\subseteq K^{(2)}\subseteq\ \ldots\ \subseteq K=\overline{\cup\, K^{(k)}}, \end{aligned} \tag{19}$$

such that for $k=1, 2, \ldots$ and $F^{(k)}:=P+C^{(k)}$,

$$F^{(k)}\cap K^{(k)}\neq\emptyset, \tag{20}$$

and

$$L_F\subseteq C^{(k)}\subseteq L_G^{\dagger},\ L_F\subseteq K^{(k)}\subseteq L_G^{\dagger}. \tag{21}$$

We construct such a sequence of cones $C^{(k)}$ from a sequence $\{X^{(k)}\}_{k=1,2,\ldots}$ of points everywhere dense in C. Since $F\cap K\neq\emptyset$, we may assume that $X^{(1)}+P\epsilon F\cup K$ and put

$$C^{(k)}:=\mathscr{C}\{X^{(1)},\ \ldots,\ X^{(k)}\}+L_F$$

Both $\mathscr{C}\{X^{(1)},\ \ldots,\ X^{(k)}\}$ and L_F are polyhedral cones, and the sum of two polyhedral cones is again polyhedral. Sequence $K^{(k)}$ is formed similarly. $C^{(k)}\cap K^{(k)}\subseteq C\cap K\subseteq L_G^{\dagger}$ by (4). Thus (21) is satisfied.

The sequences of the polar cones $C^{(k)p}$ and $-K^{(k)p}$ are decreasing sequences which approximate C^p and $-K^p$, respectively,

$$\begin{aligned} C^{(1)p}\supseteq C^{(2)p}\supseteq\ \ldots\ \supseteq C^p=\cap\, C^{(k)p}\\ -K^{(1)p}\supseteq -K^{(2)p}\supseteq\ \ldots\ \supseteq -K^p=\cap\, -K^{(k)p}, \end{aligned} \tag{22}$$

and satisfy for $k=1, 2, \ldots$ and $G^{(k)}:=Q+C^{(k)p}$,

$$G^{(k)}\cap(-K^{(k)p})\neq\emptyset \tag{23}$$

and

$$L_G\subseteq C^{(k)p}\subseteq L_F^{\dagger}\ \textit{and}\ L_G\subseteq -K^{(k)p}\subseteq L_F^{\dagger}. \tag{24}$$

Indeed if $X\epsilon C$, then there exists a sequence $\{X^{(k)}\}_{k=1,2,\ldots}$ such that $X^{(k)}\epsilon C^{(k)}$ and $X^{(k)}\to X$. For every $Y\epsilon\cap\, C^{(k)p}$ and all k we have therefore $Y^TX^{(k)}\leq 0$. Hence $Y^TX\leq 0$ for all $X\epsilon C$, which proves $Y\epsilon C^p$. The inclusion $C^p\subseteq\cap\, C^{(k)p}$ is trivial. The same argument gives $-K^p=\cap\, -K^{(k)p}$. The properties (23) and (24) follow immediately from $G^{(k)}\supseteq G$, $-K^{(k)p}\supseteq -K^p$, and $L_F^{\dagger}\subseteq C^{(k)}$, $-K^{(k)}\subseteq L_G^{\dagger}$ from polarizing (21).

In what follows, we will replace sequences (19), and thereby sequences (22), repeatedly by suitable subsequences. The terms "sequences (19)" and "sequences (22)" will always refer to the *current*, and not to the original specimens. Similarly, $C^{(k)}$, $K^{(k)}$ will be the kth elements of the sequences (19) as they stand *after* modifications, and $U^{(k)}\epsilon K^{(k)}$, $V^{(k)}\epsilon -K^{(k)p}$ will always be orthogonal solutions of $F^{(k)}=P+C^{(k)}$, $G^{(k)}=Q+C^{(k)p}$. Such orthogonal solutions exist by *Theorem P*.

The orthogonal solutions $U^{(k)}$, $V^{(k)}$ are in general not uniquely determined. They can be modified by adding arbitrary directions $D \epsilon L_F$, $E \epsilon L_G$ respectively. Indeed $D \epsilon L_F$, $E \epsilon L_G$ are directions of infinity of $F^{(k)}$, $G^{(k)}$ with respect to $K^{(k)}$, and we have seen that adding such multiples leads to new orthogonal solutions. Now let $D_1, \ldots, D_s$ be a basis of L_F. Then there exist multipliers α_i such that $\bar{U}^{(k)} := U^{(k)} - \Sigma \alpha_i D_i \epsilon L_F$. It follows that without restriction of generality, we may assume

$$U^{(k)} \epsilon L_F^{\perp} \text{ and } V^{(k)} \epsilon L_G^{\perp}. \tag{25}$$

Now either sequences (19) can be replaced by subsequences such that the limits

$$U = \lim U^{(k)}, \; V = \lim V^{(k)}$$

exist, or $||U^{(k)}|| \to \infty$ or $||V^{(k)}|| \to \infty$.

Assume that $||U^{(k)}|| \to \infty$. If sequences (19) are replaced by a suitable subsequence, then the sequence $\{U^{(k)}/||U^{(k)}||\}_{k=1, 2, \ldots}$ converges towards a direction D, $||D|| = 1$. We have

$$D = \lim \frac{U^{(k)} - P}{||U^{(k)} - P||} = \lim \frac{U^{(k)}}{||U^{(k)}||}.$$

As C and K are closed by hypothesis,

$$D \epsilon K \cap C. \tag{26}$$

Select any point $\bar{U} \epsilon F^{(1)} \cap K$. Then $\bar{U} \epsilon F^{(k)} \cap K^{(k)}$, and since $U^{(k)}$ is an optimal solution of program (8) over $F^{(k)} \cap K^{(k)}$, we have

$$U^{(k)T} Q \leq \bar{U}^T Q \text{ for all } k.$$

In other words, the sequence $\{U^{(k)T} Q\}_{k=1, 2, \ldots}$ is bounded above, whence

$$D^T Q = \lim \frac{U^{(k)T} Q}{||U^{(k)}||} \leq 0. \tag{27}$$

Select any point $\bar{V} \epsilon G \cap (-K^p)$. Then $D^T \bar{V} \geq 0$ as $D \epsilon K$ and $\bar{V} \epsilon - K^p$, and $D^T (\bar{V} - Q) \leq 0$ as $D \epsilon C$ and $\bar{V} - Q \epsilon C^p$. Hence $D^T Q \geq 0$. Together with (27), this gives

$$D^T Q = 0.$$

In other words, D is a direction of infinity of F, and

$$D \epsilon L_F$$

by (13). On the other hand, $D \epsilon L_F^{\perp}$ by (25). Thus $D = 0$, which contradicts $||D|| = 1$. Therefore $||U^{(k)}|| \to \infty$ cannot hold.

Assume then $||V^{(k)}|| \to \infty$. For suitable subsequences of sequences (23),

$$E = \lim \frac{V^{(k)} - Q}{||V^{(k)} - Q||} = \lim \frac{V^{(k)}}{||V^{(k)}||} \epsilon C^p \cap (-K^p) \cap L_G$$

and $||E|| = 1$. Select any point $\bar{V} \epsilon G \cap (-K^p)$. Then $\bar{V} \epsilon G^{(k)} \cap (-K^{(k)p})$ as sequences (22) are decreasing. Since $V^{(k)}$ is an optimal solution of program (9),

$$P^T V^{(k)} \leq P^T \bar{V} \text{ for all } k,$$

and therefore

$$P^T E = \lim \frac{P^T V^{(k)}}{||V^{(k)}||} \leq 0.$$

It then follows as before that $E \epsilon L_G$, contradicting $||E|| = 1$ and $E \epsilon L_G^{\perp}$.

We may therefore assume that limits U, V exist. $U \epsilon F \cap K$ and $V \epsilon G \cap (-K^p)$, as C and K are closed. $U^{(k)T} V^{(k)} = (U^{(k)} - P)^T (V^{(k)} - Q) = 0$ for all k by orthogonality, and this carries over to U and V. The latter are therefore orthogonal solutions of F and G. This completes the proof of *Theorem S*.

5. Fenchel Duality

Fenchel [2, 3] considers the following pair of dual programs

$$\begin{aligned} &\textit{Minimize } f(X) - g(X) \\ &\textit{Maximize } g^c(Y) - f^c(Y). \end{aligned} \tag{28}$$

Here f is a convex function $f|R^n \to R \cup \{+\infty\}$ and g is a concave function $g|R^n \to R \cup \{-\infty\}$. It is convenient to consider f and g as defined everywhere on R^n and to admit infinite function values. The obvious interpretation is used, whenever infinite values occur in the convexity (concavity) conditions:

$$f(\lambda_1 X_1 + \lambda_2 X_2) \leq \lambda_1 f(X_1) + \lambda_2 f(X_2)$$

$$g(\lambda_1 X_1 + \lambda_2 X_2) \geq \lambda_1 g(X_1) + \lambda_2 g(X_2)$$

for all $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$. The following "domains of finiteness"

$$K(f) := \{X \epsilon R^n | f(X) < +\infty\},\ K(g) := \{X \epsilon R^n | g(X) > -\infty\}$$

are convex under these circumstances.

The function f^c is the "convex conjugate" of f, and the function g^c the "concave conjugate" of g. They are defined by

$$f^c(Y) := \sup_X (Y^T X - f(X)),\ g^c(Y) := \inf_X \{Y^T X - g(X)\}.$$

From this definition,

$$f^c(Y) \geq Y^T X - f(X),\ g^c(Y) \leq Y^T X - g(X)$$

and therefore

$$f(X) - g(X) \geq g^c(Y) - f^c(Y)$$

for arbitrary X, $Y \epsilon R^n$. In other words,

$$\inf_X (f(X) - g(X)) \geq \sup_Y (g^c(Y) - f^c(Y)).$$

The question examined by Daulity Theory is under which circumstances inf = sup, or stronger, min = max. The conditions

$$K(f) \cap K(g) \neq \emptyset \textit{ and } K(f^c) \cap K(g^c) \neq \emptyset \tag{29}$$

are obviously necessary for min = max to hold. Another, in general necessary, condition is that the functions

$$\text{f } \textit{and} \text{ g } \textit{are "closed"}, \tag{30}$$

that is, lower semicontinuous convex and upper semicontinuous concave, respectively. In this case, $f^{cc}=f$ and $g^{cc}=g$.

Except for polyhedral functions, hypotheses (29) and (30) are not sufficient for min = max, not even for inf = sup to hold. So-called "duality gaps" may occur, as first communicated by Stoer in a letter to Karlin. The stronger hypotheses,

$$K(f)^I \cap K(g)^I \neq \emptyset \text{ and } K(f^c)^I \cap K(g^c)^I \neq \emptyset, \tag{31}$$

together with (30) ensure min = max in the general case. Rockafellar [5] and Stoer [6, 7] investigated "stability" hypotheses weaker than (31) but stronger than (29), for which min = max or inf = sup obtains.

This situation is analogous to the one examined in this paper: *Theorem P* holds for polyhedral cones but not in general. *Theorem I* holds in general but is too weak to yield *Theorem P* in the special case of polyhedral cones. *Theorem S* finally gives a general result subject to stability conditions. One is therefore lead to expect a relationship between Fenchel's Duality theorem and *Theorem S*. We proceed to show that they are in fact equivalent. More precisely, we prove:

THEOREM E: *For each triple* $F = P + C, G = Q + C^p, K$, *where* C, K *are arbitrary closed cones and* P, Q *arbitrary points, there exist closed functions* f *and* g, *convex and concave respectively, such that* F, G *have orthogonal solutions* $U \epsilon K$, $V \epsilon - K^p$ *if and only if min = max for the corresponding programs (28), and vice versa, for each pair of such closed functions* f, g *there exists a triple* F, G, K *such that again equivalence holds between the statement min = max and the existence of orthogonal solutions.*

PROOF: Given a triple $(F=P+C, \quad G=Q+C^p, K)$, define

$$f(X) := \begin{cases} Q^T X \text{ for } X \epsilon K \\ \infty \text{ otherwise} \end{cases}$$

$$g(X) := - \begin{cases} 0 \text{ for } X \epsilon P + C \\ -\infty \text{ otherwise.} \end{cases}$$

Then

$$f^c(Y) = \sup_X (Y^T X - f(X)) = \sup_{X \epsilon K} (Y-Q)^T X = \begin{cases} 0 \text{ for } Y \epsilon Q + K^p \\ \infty \text{ otherwise.} \end{cases}$$

Putting $Z := X - P$, one has similarly

$$g^c(Y) = \inf_X (Y^T X - g(X)) = \inf_{X \epsilon P + C} Y^T X = Y^T P + \inf_{Z \epsilon C} Y^T Z,$$

and therefore

$$g^c(Y) = \begin{cases} Y^T P \text{ if } Y \epsilon - C^p \\ -\infty \text{ otherwise.} \end{cases}$$

Programs (28) then become

Minimize $Q^T X$ *with* X *subject to*

$$X \epsilon (P + C) \cap K$$

Maximize $Y^T P$ *with* Y *subject to*

$$Y \epsilon (Q + K^p) \cap (-C^p)$$

Substituting $U := X$, $V := Q - Y$, and transforming the maximization program into a minimization program yields programs (8) and (9), and it has been shown that the existence of orthogonal solutions is equivalent to min + min = 0 for these programs.

In order to prove the converse direction, we transform the minimization program (28) into

$$\textit{Minimize } z_1 - z_2 \textit{ with } z_1, z_2, X \textit{ subject to} \tag{32}$$

$$z_1 \geqslant f(X),\ z_2 \leqslant g(X).$$

Introducing the cones

$$C_f := \mathscr{C}\left[\begin{pmatrix} X \\ z \\ 1 \end{pmatrix} | z \geqslant f(X)\right]$$

$$C_g := \mathscr{C}\left[\begin{pmatrix} X \\ z \\ 1 \end{pmatrix} | z \leqslant g(X)\right],$$

and the linear subspace

$$L := \left\{\begin{bmatrix} X \\ z_1 \\ 0 \\ X \\ z_2 \\ 0 \end{bmatrix} | z_1, z_2 \epsilon R, X \epsilon R^n\right\},$$

we formulate program

$$\textit{Minimize } z_1 - z_2 \textit{ subject to} \tag{33}$$

$$\begin{bmatrix} X_1 \\ z_1 \\ t_1 \\ X_2 \\ z_2 \\ t_2 \end{bmatrix} \epsilon (\bar{C}_f \times \bar{C}_g) \cap \left[\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + L\right]. \tag{34}$$

We proceed to prove, that

$$\textit{programs (32) and (33) are equivalent.} \tag{35}$$

We have to show, that if (z_1, z_2, X) is a feasible solution of (32), then $(t_1 = 1, t_2 = 1, z_1, z_2, X_1 = X, X_2 = X)$ is a feasible solution of (33), and vice versa.

For an auxiliary argument, consider any closed convex set K in R^k, and the convex cone

$$H(K) := \mathscr{C}\left\{\begin{pmatrix} W \\ 1 \end{pmatrix} \middle| W \epsilon K\right\}$$

Then $H(K)$ is not necessarily closed. However, $\overline{H(K)}$ and $H(K)$ differ only in points $\begin{pmatrix} W \\ w \end{pmatrix}$ with $w = 0$. Indeed, let

$$\begin{pmatrix} W^{(n)} \\ w^{(n)} \end{pmatrix} \to \begin{pmatrix} W \\ w \end{pmatrix}$$

be a convergent sequence of nonzero elements of $H(K)$, i.e., $W^{(n)}/w^{(n)} \epsilon K$. If $w > 0$, then

$W^{(n)}/w^{(n)} \to W/w \epsilon K$ as K is closed. But then $\begin{pmatrix} W \\ w \end{pmatrix} \epsilon H(K)$. Thus only if $w=0$ may $\begin{pmatrix} W \\ w \end{pmatrix}$ be in $\overline{H(K)}$ but not in $H(K)$.

As the functions f and g are closed, so are the convex sets

$$\left\{ \begin{pmatrix} X_1 \\ z_1 \end{pmatrix} | z_1 \geq f(X_1) \right\}, \left\{ \begin{pmatrix} X_2 \\ z_2 \end{pmatrix} | z_2 \leq g(X_2) \right\}.$$

The above argument can therefore be employed to show that

$$(\bar{C}_f \times \bar{C}_g) \cap T = (C_f \times C_g) \cap T,$$

where T denotes the plane characterized by $t_1 = t_2 = 1$. In the expression (34) for the feasible region of (33), one can then replace $(\bar{C}_f \times \bar{C}_g)$ by $(\bar{C}_f \times \bar{C}_g) \cap T$, $(C_f \times C_g) \cap T$ and $(C_f \times C_g)$ successively, without changing the set. It is then plain that the points in (34) correspond to the feasible solutions of (32).

Program (33) *is of type* (8).

The product $\bar{C}_f \times \bar{C}_g$ plays the role of K, and the linear subspace L plays the role of C. Furthermore

$$Q = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, P = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

We have to show, that the corresponding program (9),

Minimize $w_1 + w_2$ *subject to* (36)

$$\begin{bmatrix} Y_1 \\ s_1 \\ w_1 \\ Y_2 \\ s_2 \\ w_2 \end{bmatrix} \epsilon - (\bar{C}_f \times \bar{C}_g)^p \cap \left[\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} + L^\perp \right]$$

is indeed equivalent to the maximization program (28). To this end, we note that

$$L^\perp = \left\{ \begin{bmatrix} -Y \\ 0 \\ w_1 \\ Y \\ 0 \\ w_2 \end{bmatrix} | w_1, w_2 \epsilon R, Y \epsilon R^n \right\}.$$

All feasible solutions of program (36) are therefore necessarily of the form

$$\begin{bmatrix} -Y \\ 1 \\ w_1 \\ Y \\ -1 \\ w_2 \end{bmatrix}.$$

These points must moreover belong to $-(\bar{C}_f \times \bar{C}_g)^p$. As $(\bar{C}_f \times \bar{C}_g)^p = (\bar{C}_f \times \bar{C}_g)^p = (C_f \times C_g)^p$, all feasible solutions are characterized by

$$-Y^T X + z_1 + w_1 \geq 0 \text{ and } Y^T X - z_2 + w_2 \geq 0 \textit{ whenever } z_1 \geq f(X) \textit{ and } z_2 \leq g(X),$$

in addition to being of the form (37). These conditions are clearly equivalent to

$$-Y^T X + f(X) + w_1 \geq 0 \textit{ for all } X \epsilon R^n$$

and

$$Y^T X - g(X) + w_2 \geq 0 \textit{ for all } X \epsilon R^n,$$

respectively. These in turn are equivalent to

$$w_1 \geq f^c(Y) \textit{ and } w_2 \geq -g^c(Y).$$

Hence program (36) becomes

$$\textit{Minimize } f^c(Y) - g^c(Y)$$

or

$$\textit{Maximize } g^c(Y) - f^c(Y),$$

which was to be shown.—

The main part of this work was carried out at the National Bureau of Standards in Washington, D.C., where the author benefited from the interest and encouragement of A. J. Goldman. The author acknowledges the thorough and detailed examination of the manuscript by the referee of this Journal.

6. References

[1] Tucker, A. W., Abstract Structure of the Simplex Method, The RAND Symposium on Linear Programming, The RAND Corporation, Report R-351, June 21, 1959, pp. 33–34.
[2] Fenchel, W., On conjugate convex functions, Canad. J. Math. **1**, 73–77 (1949).
[3] Fenchel, W., Convex cones, sets and functions, Lecture notes by D. W. Blackett, Princeton University 1953.
[4] Kretschmer, K. S., Programs in paired spaces, Canad. J. Math. **13**, 221–238 (1961).
[5] Rockafellar, R. T., Convex functions and dual extremum problems, Diss. Harvard Univ., Cambridge, Mass. (1963).
[6] Stoer, J., Duality in convex programming, Habilitationsschrift. Technische Hochschule München (1964).
[7] Stoer, J., Duality in nonlinear programming and the minimax theorem. Numer. Math. **5**, 371–379 (1963).

(Paper 74B2–322)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 74B, No. 2, April–June 1970

On Contractive Semigroups and Uniform Asymptotic Stability

Philip R. Meyers*

**Institute for Basic Standards, National Bureau of Standards,
Washington, D. C. 20234**

(March 31, 1970)

This paper calls attention to the equivalence between two well-known mathematical ideas: contraction mappings (in the sense of Banach) and asymptotic stability. The equivalence is formalized by defining a flow (representing the possible movements over time of some system through its state space) as a continuous one-parameter semigroup of operators on a metric space, and then showing that these operators are all contractions (in suitably revised metrics) if and only if there is a uniformly asymptotically stable equilibrium point. Generalizations to other operator semigroups are also given.

Key words: Contractions; control theory; functional analysis; semigroup; stability theory; topology.

1. Introduction

Our aim is to point out, and to make explicit, the relationship between two well-known mathematical ideas: contraction mappings (in the sense of Banach) and asymptotic stability. In this preliminary section we collect the relevant definitions and background material, and state our principal results. The proofs of these results are given in the following section, and various generalizations are given in the concluding part.

Let (X, d) be a metric space, and recall that a map $f: X \to X$ is called a *contraction* of (X, d) if there exists $\lambda \in (0, 1)$, known as a *contraction constant* for f, such that

$$d(f(x), f(y)) \leq \lambda d(x, y)$$

for all $x, y \in X$. (Such an f is necessarily continuous.) For present purposes, the Banach Contraction Principle can be conveniently stated as follows:

THEOREM A: *Let* f *be a contraction of a complete*[1] *metric space* (X, d). *Then there exists* $\xi \in X$ *such that*

(i) $f(\xi) = \xi$,

(ii) $f^n(x) \to \xi$ *for each* $x \in X$,

(iii) $f^n(U) \to \{\xi\}$ *for some neighborhood* U *of* ξ.

It is not difficult to see a connection between conditions (i), (ii), and (iii), and the definition of asymptotic stability. This connection is formalized below. The main tool is the following converse to Theorem A, proved by the author in [1]:[2]

THEOREM B: *Let* f *be a continuous self-mapping of a metrizable topological space* X, *which for some* $\xi \in X$ *satisfies the three conditions of Theorem A. Then for each* $\lambda \in (0, 1)$ *there exists a metric*

*Present address: IBM World Trade Corporation, 821 United Nations Plaza, New York, New York 10017.

[1] The completeness hypothesis is inessential, but its omission would require more complicated statements involving equivalent Cauchy sequences.

[2] Figures in brackets indicate the literature references at the end of this paper.

382-233 O - 70 - 4

d_λ, *which is compatible with the topology of* X *and can be chosen complete if* X *admits a complete metric, such that* f *is a contraction of* (X, d_λ) *with contraction constant* λ.

Now let (X, d) be a metric space, and S a commutative topological semigroup (written additively) with zero element. Then a family $\{T_s : s \in S\}$ of continuous self-mappings of X (not necessarily homeomorphisms) will be called an *S-semigroup of operators* on X if it obeys

$$T_0(x) = x \qquad x \in X,$$

the semigroup condition

$$T_s(T_t(x)) = T_{s+t}(x) \qquad s, t \in S;\ x \in X,$$

and the continuity condition that for each $t \in S$,

$$\sup \{d(T_t(x), T_s(x)) : x \in X\} \to 0 \quad \text{as } s \to t. \tag{1.1}$$

The semigroup of operators will be called *contractive* if for some $\lambda \in (0, 1)$ there is a family $\{d_s : s \in S\}$ of metrics of X, each giving the same topology as that of (X, d), such that T_s is a contraction of (X, d_s) with contraction constant λ for each $s \in S - \{0\}$. (It is natural to ask whether d_s can be chosen independent of s. Such questions are addressed in [2] and [3].)

In the motivating situations X is the space of possible "states" of a system whose evolution is "stationary" or "autonomous" (in the differential-equation case, $\dot{x} = f(x)$ rather than $\dot{x} = f(x, t)$), while S is a subsemigroup of the additive semigroup R^+ of nonnegative real numbers. Thus we have the interpretation

$$T_s(x) = \text{state of system at time } s \geqslant 0$$

if x is its initial state.

THEOREM 1: *For an* R^+*-semigroup* $\{T_t : t \in R^+\}$ *of operators on* X *to be contractive, it suffices (and is clearly necessary) that some one* T_t *obeys the three conditions of Theorem A.*

Extensions to more general S will be given in the final section. We turn now to the definition of asymptotic stability. In view of the interpretation of $T_s(x)$ given above, it is natural to call $\xi \in X$ an *equilibrium state* if it is a common fixed point of the semigroup of operators, i.e.,

$$T_s(\xi) = \xi \qquad \text{for } s \in S.$$

For this to be true, it suffices that some T_t have ξ as *unique* fixed point, since then for any $s \in S$

$$T_s(\xi) = T_s(T_t(\xi)) = T_{s+t}(\xi) = T_{t+s}(\xi) = T_t(T_s(\xi))$$

so that $T_s(\xi)$ is a fixed point of T_t, implying $T_s(\xi) = \xi$ as desired. In this connection note that (i) and (ii) of Theorem A imply that ξ is the *only* fixed point of f.

Recall that an equilibrium state ξ is called *stable* if for each neighborhood V of ξ there is a neighborhood W of ξ such that

$$x \in W \text{ implies } T_s(x) \in V \quad \text{for all} \quad s \in S.$$

For the case $S \subseteq R^+$, the stable equilibrium state ξ is called *asymptotically stable* if there is a neighborhood U of ξ such that

$$x \in U \text{ implies } T_s(x) \to \xi \qquad \text{as } s \to \infty,$$

and is called *uniformly* asymptotically stable if U can be so chosen that

$$T_s(U) \to \{\xi\} \qquad \text{as } s \to \infty.$$

The relation between this concept and that of a contraction mapping is given in the second of our principal results:

THEOREM 2: *The equilibrium state ξ of an R^+-semigroup of operators is uniformly asymptotically stable if and only if ξ has a neighborhood $X_0 \subseteq X$ such that the restriction of the semigroup to X_0 is a contractive semigroup.*

In many applications ξ will necessarily have a compact neighborhood (e.g., when X is a finite-dimensional Euclidean space). It will be apparent from the proofs that for such cases the continuity condition (1.1) can be weakened to a requirement that, for each compact subset C of X and each $t \epsilon S$,

$$\sup \{d(T_t(x), T_s(x)) : x \epsilon C\} \to 0 \qquad \text{as } s \to t.$$

2. Proofs of Theorems 1 and 2

Throughout this section $\{T_s : s \epsilon R^+\}$ is an R^+-semigroup of operators on metric space (X, d), and for $x \epsilon X$ and $r > 0$ we set

$$S(x, r) = \{y : y \epsilon X, d(x, y) \leq r\}.$$

The proofs of theorems 1 and 2 both use the following lemma.

LEMMA: *Let $\xi \epsilon X$ be an equilibrium state. For each $t > 0$ and $\eta > 0$, there is a $\delta > 0$ such that*

$$T_s(S(\xi, \delta)) \subseteq S(\xi, \eta) \qquad \text{for } 0 \leq s \leq t.$$

PROOF: If not, there is a $t > 0$, an $\eta > 0$, a positive sequence $\delta_n \to 0$, and associated sequences of points $x_n \epsilon S(\xi, \delta_n)$ and of numbers $s(n) \epsilon [0, t]$ such that

$$T_{s(n)}(x_n) \epsilon X - S(\xi, \eta).$$

By passing to a subsequence if necessary, we may suppose that $s(n) \to s \epsilon [0, t]$. Then

$$d(\xi, T_{s(n)}(x_n)) \leq d(T_s(\xi), T_s(x_n)) + d(T_s(x_n), T_{s(n)}(x_n)).$$

The limit as $n \to \infty$ of the first term on the right vanishes because $x_n \to \xi$, while the continuity of the semigroup of operators ensures the same for the second term. So we have a contradiction, completing the indirect proof of the lemma.

For the proof of Theorem 1, it suffices (by Theorem B) to assume that some T_t obeys conditions (i), (ii), and (iii) of Theorem A, and to show that any T_s $(s > 0)$ does the same. It was shown below the statement of Theorem 1 that T_s satisfies (i), with the same fixed point ξ as for T_t. Note that $t > 0$, since T_0 cannot satisfy (ii) . . . except for the trivial case $X = \{\xi\}$ when the theorem is true anyhow.

Condition (ii) for T_s is verified by applying the lemma in the following way: Consider any $x \epsilon X$ and any $\eta > 0$. By (ii) for T_t, we can choose $N > 0$ so large that

$$d(T_t^n(x), \xi) < \delta, \qquad \text{for } n \geq N,$$

where δ is related to t and η as in the lemma. Next choose $M > 0$ so large that

$$ms \geq Nt \qquad \text{for } m \geq M.$$

Then for $m \geq M$ we have

$$ms = nt + \sigma \qquad (n \geq N, 0 \leq \sigma < t),$$

so that by the lemma

$$T_\sigma(S(\xi, \delta)) \subseteq S(\xi, \eta),$$

while

$$T_t^m(x) = T_{sm}(x) = T_\sigma(T_t^n(x)) \; \epsilon T_\sigma(S(\xi, \delta)),$$

hence for $m \geq M = M(\eta)$, $T_t^m(x) \in S(\xi, \eta)$, proving (ii).

To prove (iii) for T_s, consider any $\eta > 0$, let δ be related to t and η as in the lemma, and (since (iii) holds for T_t) let U be a neighborhood of ξ such that $T_t^n(U) \to \{\xi\}$. Choose $N > 0$ so large that

$$T_t^n(U) \subseteq S(\xi, \delta) \qquad \text{for } n \geq N,$$

and M so large that $ms > Nt$ for $m \geq M$. Then for $m \geq M$ we can proceed as above to prove $T_s^m(U) \subseteq S(\xi, \eta)$, completing the proof.

We turn now to the first half of the proof of Theorem 2. Assuming the existence of a neighborhood X_0 of the equilibrium state ξ such that the restriction of the semigroup to X_0 is contractive, we wish to show that ξ is uniformly asymptotically stable.

Let V be any neighborhood of ξ, consider any $t > 0$, and choose $\eta > 0$ such that

$$S(\xi, \eta) \subseteq V \cap X_0.$$

Then choose δ related to t and η as in the lemma, implying $\delta \leq \eta$. There is a metric d_t on X_0, yielding the same topology as d, such that $T_t|X_0$ is a contraction on (X_0, d_t). Thus there is a $\Delta > 0$ such that

$$W = \{y : y \in X_0,\ d_t(y, \xi) < \Delta\} \subseteq S(\xi, \delta)$$

and also such that $T_t(W) \subseteq W$. Then for any $s \geq 0$, written

$$s = nt + \sigma \qquad (n \geq 0,\ 0 \leq \sigma < t),$$

we have

$$T_s(W) = T_\sigma(T_t^n(W)) \subseteq T_\sigma(W) \subseteq T_\sigma(S(\xi, \delta)) \subseteq S(\xi, \eta) \subseteq V,$$

proving that ξ is stable.

Moreover, there is a neighborhood U of ξ (independent of V) such that $U \subseteq X_0$, $T_t(U) \subseteq X_0$, and $T_t^n(U) \to \{\xi\}$. Given V, choose η and δ as above. Pick $N > 0$ so large that

$$T_t^n(U) \subseteq S(\xi, \delta) \qquad \text{for } n \geq N.$$

Then any $s \geq Nt$ can be written

$$s = nt + \sigma \qquad (n \geq N,\ 0 \leq \sigma < t)$$

so that

$$T_s(U) = T_\sigma(T_t^n(U)) \subseteq T_\sigma(S(\xi, \delta)) \subseteq S(\xi, \eta) \subseteq V.$$

Hence $T_s(U) \to \{\xi\}$ as $s \to \infty$, implying that ξ is uniformly asymptotically stable.

For the converse half of Theorem 2, assume ξ is a uniformly asymptotically stable equilibrium state, and define

$$X_0 = \{x : x \in X,\ T_s(x) \to \xi \quad \text{as } s \to \infty\}.$$

The definition of asymptotic stability ensures that X_0 contains a neighborhood of ξ, and so is itself such a neighborhood. The semigroup property implies that $T_s(X_0) \subseteq X_0$ for each $s \in R^+$, so that $\{T_s|X_0 : s \in R^+\}$ is an R^+-semigroup of operators on X_0.

Choose any $t > 0$. The definition of X_0 ensures that $T_t|X_0$ satisfies conditions (i) and (ii) of Theorem A, so only (iii) need be verified. The uniform asymptotic stability of ξ ensures the existence of a neighborhood U (in X_0, and hence in X) of ξ such that $T_s(U) \to \{\xi\}$ as $s \to \infty$, and hence in particular such that $T_t^n(U) = T_{nt}(U) \to \{\xi\}$. So (iii) also holds, and an appeal to Theorem B completes the proof of Theorem 2.

3. Remarks and Generalizations

Our first group of remarks pertain to extending Theorem 1 from $S=R^+$ to other semigroups. We begin with the *discrete* case, i.e., S is algebraically arbitrary but has the discrete topology (more precisely, the topology of S is irrelevant). The simplest instance is that in which S is the semigroup I^+ of nonnegative integers, a situation arising for example in connection with sampled-data systems. The next theorem generalizes a result, for $S=I^+$, proved in [1] as a simple corollary to Theorem B. Its statement uses the fact that the definition of "contractive," introduced earlier only for semigroups of operators on (X, d), is equally applicable to any family of self-mappings of (X, d).

THEOREM 3: *The semigroup* $\{T_s : s\epsilon S\}$ *of operators is contractive if and only if there is a generating subset* T *of* S *such that* $\{T_t : t\epsilon T\}$ *is contractive.*

PROOF: The "only if" direction is obvious; just take $T=S$. Suppose now that $\{T_t : t\epsilon T\}$ is contractive, where T generates S. Consider any $s\epsilon S-\{0\}$. Since T is a generating set, we have

$$s=t(1)+t(2)+\ldots+t(m)$$

where each $t(j)\epsilon T-\{0\}$. Thus, with composition of operators written multiplicatively, we have

$$T_s=T_{t(1)}T_{t(2)}\ldots T_{t(m)}.$$

Each $T_{t(j)}$ has a unique fixed point, and since they commute they have the same fixed point ξ. Thus T_s has ξ as fixed point, and so satisfies condition (i) of Theorem A.

To each $T_{t(j)}$ there corresponds a neighborhood U_j of ξ such that $T^n_{t(j)}(U_j)\to\{\xi\}$. As shown in the proof of Theorem B in [1], U_j can be assumed chosen so that $T_{t(j)}(U_j)\subseteq U_j$. Let $U=\bigcap_1^m U_j$ and note that, by commutativity,

$$T^n_s=T^n_{t(1)}T^n_{t(2)}\ldots T^n_{t(m)}.$$

Also, let V be any neighborhood of ξ.

Consider any $x\epsilon X$. Choose N so large that, for $1\leq j\leq m$,

(a) $T^n_{t(j)}(U_j)\subseteq U\cap V\subseteq U_j$, for $n\geq N$,

(b) $T^n_{t(m)}(x)\epsilon U\cap V\subseteq U_{m-1}$, for $n\geq N$.

Then for $n\geq N$,

$$T^n_s(x)\epsilon T^n_{t(1)}\ T^n_{t(2)}\ \ldots\ T^n_{t(m-1)}\ (U\cap V)\subseteq\ldots\subseteq T^n_{t(1)}\ (U\cap V)\subseteq U\cap V\subseteq V,$$

proving that $T^n_s(x)\to\xi$.

If N is chosen so large that (a) holds for $1\leq j\leq m$, then the same inclusions show that $T^n_s(U)\subseteq V$ for $n\geq N$, so that $T^n_s(U)\to\{\xi\}$ and an application of Theorem B completes the proof.

For the case $S=I^+$, where T can be taken to have a single element, there is clearly an analog of Theorem 2 relating contractiveness to uniform asymptotic stability.

The next case treated is essentially an amalgam of the discrete situation and the case $S=R^+$.

THEOREM 4: *Let* S *be a convex cone, with* 0 *as apex, in a real topological vector space. Then an* S*-semigroup* $\{T_s : s\epsilon S\}$ *of operators is contractive if and only if there is a subset* T *of* S *such that* $\{T_t : t\epsilon T\}$ *is contractive and each* $s\epsilon S$ *is a nonnegative finite linear combination of members of* T.

PROOF: As before, the choice $T=S$ proves the "only if". Now suppose there is a subset T with the specified properties. Consider any $s\epsilon S-\{0\}$; we have

$$s=a_1t(1)+\ldots+a_mt(m)$$

where $a_j \epsilon R^+ - \{0\}$ and $t(j) \epsilon T$ for $1 \leq j \leq m$. Define a subset Σ of S by

$$\Sigma = \{c_1 t(1) + \ldots + c_m t(m) : 0 \leq c_j \leq 1 \quad \text{for} \quad 1 \leq j \leq m\},$$

so that $0 \epsilon \Sigma$ and Σ is sequentially compact. From this compactness it is easy to verify the following analog of the lemma proved in Section 2: For any $\eta > 0$ there is a $\delta > 0$ such that

$$T_\sigma(S(\xi, \delta)) \subseteq S(\xi, \eta) \quad \text{for all} \quad \sigma \epsilon \Sigma,$$

where ξ is the equilibrium state.

For each integer $n > 0$, let $n(j)$ be the largest integer not greater than na_j, so that $n \to \infty$ implies $n(j) \to \infty$ for $1 \leq j \leq m$. Then

$$T_s^n = T_{\sigma(n)} T_{t(1)}^{n(1)} \cdot \ldots \cdot T_{t(m)}^{n(m)},$$

where $\sigma(n) \epsilon \Sigma$.

Choose neighborhoods U_j of ξ such that $T_{t(j)}^n (U_j) \to \{\xi\}$, $T_{t(j)}(U_j) \subseteq U_j$, and $U_j \subset S(\xi, \delta)$. Then the proof of Theorem 3 is readily modified to show, for any $\eta > 0$, that for any $x \epsilon X$,

$$T_s^n (x) \epsilon T_{\sigma(n)}(S(\xi, \delta)) \subseteq S(\xi, \eta)$$

for all large enough n, and that for $U = \bigcap_1^m U_j$,

$$T_s^n(U) \subseteq T_{\sigma(n)}(S(\xi, \delta)) \subseteq S(\xi, \eta)$$

for all large enough n. This and an application of Theorem B complete the proof.

There is no exploration of analogous extensions of Theorem 2, since the author has not found an intuitively satisfying interpretation of the concept $s \to \infty$ so essential in stability considerations.

4. References

[1] Meyers, P. R., A Converse to the Banach Contraction Theorem, Nat. Bur. Stand. (U.S.), 71B (Math. and Math. Phys.), Nos. 1 & 2, 73–76 (Apr.–Sept. 1967).

[2] Goldman, A. J., and Meyers, P. R., Simultaneous Contractification, Nat. Bur. Stand. (U.S.), 73B (Math. Sci.), No. 4, 301–305 (Oct.–Dec. 1969).

[3] Meyers, P. R., Contractifiable Semigroups and Lyapanov Functions, in preparation.

(Paper 74B2–323)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 74B, No. 2, April–June 1970

Normal Subgroups of the Modular Group*

Leon Greenberg and Morris Newman**

**Institute for Basic Standards, National Bureau of Standards,
Washington, D.C. 20234**

(July 2, 1969)

A number of results on the normal subgroup structure of the classical modular group is announced. A typical result is that a normal subgroup of square-free index is necessarily of genus 1, apart from 4 exceptions.

Key words: Genus; index; modular group; normal subgroups.

1. Introduction

In this note we summarize the results of some work on the normal subgroups of the classical modular group Γ, which is a continuation of the work begun in [1][1] and [4]. We may regard Γ as the free product of a cyclic group of order 2 and a cyclic group of order 3; $\Gamma = \{x\} * \{y\}, x^2 = y^3 = 1$. The number of normal subgroups of Γ of index μ will be denoted by $N(\mu)$. If G is any subgroup of Γ, G' will denote its commutator subgroup, and G^p the fully invariant subgroup of G generated by the pth powers of the elements of G. The *level* of G is the least positive integer n such that $(xy)^n \in G$. If G is a normal subgroup of index $\mu \geq 6$ and n is its level, then the genus of G is given by

$$g = 1 + \mu(n-6)/12n,$$

and the number of parabolic classes of G by

$$t = \mu/n.$$

Except for the groups Γ, Γ^2, or Γ^3, the index of a normal subgroup is a multiple of 6.

The commutator subgroup Γ' of Γ is a free group of rank 2, freely generated by

$$a = xyxy^2, \qquad b = xy^2xy.$$

The normal subgroups of Γ of genus 1 (alternatively, of level 6) have been completely described in [5]. Any such subgroup G lies between Γ' and Γ'' and may be described uniquely by the triplet of integers (p, m, d), where $p > 0$, $0 \leq m \leq d-1$, $m^2+m+1 \equiv 0 \bmod d$. G is of index $6dp^2$ in Γ and consists of all words w of Γ' satisfying

$$e_a(w) \equiv 0 \bmod p, \qquad e_b(w) \equiv m e_a(w) \bmod dp,$$

where $e_a(w)$, $e_b(w)$ are the respective exponent sums in a and b of w.

We also let $G_{k,m}$ be the intersection of all normal subgroups of Γ containing

$$(xy)^{mk}, \qquad (yx)^k (xy)^{-k}.$$

---

*This paper was written while the first author held NSF Grant GP8919.

**Present address University of Maryland, College Park, Md. 30740.

[1] Figures in brackets indicate the literature references at the end of this paper.

Going over to the representation of Γ as $LF(2, Z)$, we define the *principal congruence subgroup* $\Gamma(n)$ as the totality of elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ such that

$$a \equiv d \equiv \pm 1 \bmod n, \qquad b \equiv c \equiv 0 \bmod n.$$

2. The Results

We now state the principal results obtained. Throughout this section G is a normal subgroup of Γ of index μ, level n, genus g, and having t parabolic classes.

(1) Suppose that μ is square-free. Then either $G = \Gamma$, Γ^2, Γ^3, or $\Gamma(2)$, or else G is of genus 1 and every prime divisor of $\mu/6$ is $\equiv 1 \bmod 3$.

(2) Define $f(\mu)$ as 1 if there is a normal subgroup G of index μ with solvable quotient group Γ/G, and 0 otherwise. Then

$$\lim_{x \to \infty} \frac{1}{x} \sum_{\mu \leq x} f(\mu) = 0.$$

(3) Let p be a prime, $p \equiv -1 \bmod 3$, and suppose that $p > r$. Then there is no normal subgroup of Γ of index pr.

(4) If Γ/G is nilpotent then it is abelian, and G must be Γ, Γ^2, Γ^3, or Γ'.

(5) Let p be a prime, $p \equiv 1 \bmod 12$. Then there are no normal subgroups of Γ having $2p$ parabolic classes.

(6) Let p be a prime > 84, $p \equiv -1 \bmod 3$; and let n be any positive integer. Then there are no normal subgroups of Γ of genus $1 + p^n$.

(7) Let p be a prime > 5, and suppose that $\mu = 6p^2$. Then G must be one of the following groups: groups:

(i) $\Gamma(2)^p\Gamma(2)'$.

(ii) $(p, 0, 1)$.

(iii) $(1, m_1, p^2)$, $(1, m_2, p^2)$, where $p \equiv 1 \bmod 3$ and m_1, m_2 are the solutions of $m^2 + m + 1 \equiv 0 \bmod p^2$.

Thus

$$N(6p^2) = 3 + (p/3), \quad p \text{ prime}, p > 5.$$

(8) Let p be a prime > 11. Then $N(12p^2) = 0$.

(9) Let p be a prime > 11. Then the only normal subgroup of Γ of index $12p^3$ is $\Gamma(3)^p\Gamma(3)'$.

(10) $N(72) = 2$, $N(78) = 2$, $N(84) = 0$, $N(90) = 0$.

(11) There is just one normal subgroup of Γ of genus 2: namely $G_{4,2}$.

(12) The normal subgroups of Γ with t parabolic clasese, $t \leq 5$, are the following:

$t = 1$: Γ, Γ^2, Γ^3, Γ'.

$t = 2$: none.

$t = 3$: $\Gamma(2)$, $(1, 1, 3)$.

$t = 4$: $(2, 0, 1)$, $\Gamma(3)$, $G_{3,4}$.

$t = 5$: none.

3. Some Remarks

Perhaps the most striking results are the first two. A generalization of (2) with a precise estimate for the density function is in course of publication ([2]). As for (1), we note that if G is any

finite group of square-free order generated by elements x, y such that $x^2=y^3=1$, then $(xy)^6=1$. This is so since the second commutator subgroup G'' is necessarily $\{1\}$ (p. 148 of [3]), and

$$a = xyxy^2 \epsilon G', \quad b = xy^2xy \epsilon G',$$
$$(xy)^6 = ab^{-1}a^{-1}b \epsilon G''.$$

The result (1) is now an easy consequence.

The other results are of varying degrees of difficulty, but generally present no special problems.

4. References

[1] Greenberg, Leon, Note on normal subgroups of the modular group, Proc. Amer. Math. Soc. **17,** 1195–1198 (1966).

[2] Greenberg, Leon, and Newman, Morris, Some results on solvable groups, Arch. Math. (to appear).

[3] Hall, Marshall, The Theory of Groups (The Macmillan Company, New York, 1959).

[4] Newman, Morris, Classification of normal subgroups of the modular group, Trans. Amer. Math. Soc. **126,** 267–277 (1967).

[5] Newman, Morris, A complete description of the normal subgroups of genus one of the modular group, Amer. J. Math. **86,** 17–24 (1964).

(Paper 74B2–324)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 74B, No. 2, April–June 1970

Publications of the National Bureau of Standards*

Selected Abstract

Lehner, J., **Lectures on modular forms,** *Nat. Bur. Stand. (U.S.), Appl. Math. Ser.* **61,** *77 pages (Dec. 1969), 70 cents, SD Catalog No. C13.32:61.*

Key words: Automorphism; finite field; genus; Hecke operator; Klein's invariant; linear transformation; modular form; Riemann surface.

This book is an expository account of the theory of modular forms and its application to number theory and analysis. The first chapter defines modular forms and develops their most important properties. The second and third chapters introduce the Hecke modular forms. A Hecke form f is a simultaneous eigenfunction of a family of linear operators T_p, where p runs over the positive primes. The eigenvalue corresponding to T_p is the pth Fourier coefficient $a(p)$ of f. The Fourier coefficients of f are multiplicative, i.e., $a(mn)=a(m)a(n)$ when m and n have no common divisors. The intimate connection of these results to the theory of certain zeta-functions is indicated in a note to chapter 3. The fourth chapter is devoted to the automorphisms of a compact Riemann surface. It is shown what groups of linear-fractional transformations with coefficients in a finite field can be the group of automorphisms of a compact Riemann surface. In the last two chapters congruences and other arithmetic properties are developed for the Fourier coefficients of Klein's absolute modular invariant. Analogies with the Hecke theory as well as with the Ramanujan congruences for the partition function are discussed.

Other NBS Publications

J. Res. Nat. Bur. Stand. (U.S.), 74A (Phys. Chem.), No. 2 (Mar.–Apr. 1970), SD Catalog No. C13.22/sec. B:74/2, $2.00.

A theroetical investigation of the configurations $(3d+4s)^4 4p$ in neutral vanadium (V I). C. Roth.

A theoretical investigation of the configurations $(3d+4s)^5 4p$ in neutral chromium (Cr I). C. Roth.

A theoretical investigation of the configurations $(3d+4s)^7 4p$ in neutral iron. C. Roth.

The intensity as a function of temperature of the low-angle x-ray diffraction maxima of the n-paraffins: hexatriacontane, tetratetracontane, and tetranonacontane. P. K. Sullivan and J. J. Weeks.

Photoelastic constants of ruby. R. M. Waxler and E. N. Farabaugh.

Estimation of critical constants T_c, ρ_c from the $\rho(T)$ and $T(\rho)$ relations at coexistence. R. D. Goodwin.

Synthesis and growth of fresnoite ($Ba_2TiSi_2O_8$) from a TiO_2 flux and its relation to the system $BaTiO_3-SiO_2$. C. R. Robbins.

Photoionization study of $Fe(CO)_5$ and $Ni(CO)_4$. G. Distefano.

Papers presented at the Conference, Electronic Density of States, National Bureau of Standards, Gaithersburg, Maryland November 3–6, 1969

The band structure problem. J. M. Ziman.

Optical properties and electronic density of states. M. Cardona.

Excitonic effects in x-ray transitions in metals. G. D. Mahan.

Soft x-ray band spectra and their relationship to the density of states. G. A. Rooke.

The electronic structure of disordered alloys. J. L. Beeby.

Summary of the conference on electronic density of states. H. Ehrenreich.

Allan, D. W., Guetrot, A., Higbie, L. S., Lavanceau, J. D., An application of statistical smoothing techniques on VLF signals for comparison of time between USNO and NBS, Proc. 23rd Frequency Control Symp., Atlantic City, N.J., May 6–8, 1969, p. 248 (1969).

Alvarez, R., Paulsen, P. J., Kelleher, D. E., Simultaneous determination of trace elements in platinum by isotope dilution and spark source mass spectrometry, Anal. Chem. **41,** No. 7, 955–958 (June 1969).

Angeles, R., Aldridge, M., Freeman, D., Wall, L., Cryodegradation of polyelectrolyte networks, Polymer Letters **7,** No. 8, 609–611 (1969).

Armstrong, G. T., Hydrogen fluoride and the thermochemistry of fluorine, Proc. Conf. Interagency Chemical Rocket Propulsion Group, Thermochemistry Working Group. Cleveland, Ohio, Apr. 9–11, 1969, 7th Meeting Bulletin 1, 119–135 (Chemical Propulsion Information Agency, Johns Hopkins University, Applied Physics Laboratory, Silver Spring, Maryland, Aug. 1969).

Armstrong, G. T., King, R. C., The heats of formation of some fluorine-containing oxidizers, Proc. Conf. Interagency Chemical Rocket Propulsion Group, Thermochemistry Working Group, Cleveland, Ohio, Apr. 9–11, 1969, 7th Meeting Bulletin 1, 19–40 (Chemical Propulsion Information Agency, Johns Hopkins University, Applied Physics Laboratory, Silver Spring, Maryland, Aug. 1969).

Bates, R. G., Measurement of pH, Handbook of Biochemistry, H. A. Sober, ed., pp. J190–J194 (Chemical Rubber Company, Cleveland, Ohio, 1967).

Beatty, R. W., Coaxial power, impedance and attenuation calibration at the National Bureau of Standards, Microwave J. **11,** No. 5, 65–75 (May 1968).

Bennett, L. H., Charge transfer in intermetallic compounds: example–the sodium thallide structure, Chapter in Structural Developments in Alloy Phases, B. C. Giessen, ed., Discussion I, 41–44 (Plenum Press, Inc., New York, N.Y., 1969).

Birmingham, B. W., Flynn, T. W., Cryogenics in United States National Programs, Cryogenics **9,** No. 1, 3–10 (Feb. 1969).

Braun, W., Bass, A. M., Davis, D. D., Simmons, J. D., Flash photolysis of carbon suboxide: absolute rate constants for reactions of $C(^3P)$ and $C(^1D)$ with H_2, N_2, CO, NO, O_2 and CH_4, Proc. Roy. Soc. **A312,** 417–434 (1969).

Brennan, J. A., Edmonds, D. K., Smith, R. V., Two-phase (liquid-vapor), mass-limiting flow with hydrogen and nitrogen (Proc. 1968 Cryogenic Engineering Conf., Case Western Reserve Univ., Cleveland, Ohio, Aug. 19–21, 1968), Chapter in Advances in Cryogenic Engineering 14, Paper G–6, 294–298 (Plenum Press, Inc., New York, N.Y., 1969).

Brown, J. E., Tryon, M., Horowitz, E., Thermal decomposition of Cu(II)-*bis*(8-hydroxy-5-quinolyl) methane coordination polymer, J. Appl. Poly. Sci. **13,** 1937–1947 (1969).

Bullis, W. M., Measurement methods for the semiconductor device industry—a summary of NBS activity, Nat. Bur. Stand. (U.S.), Tech. Note 511, 24 pages (Dec. 1969), 30 cents, SD Catalog No. C13.46:511.

Bullis, W. M., Editor, Methods of measurement for semiconductor materials, process control, and devices, Quarterly Report April 1 to June 30, 1969, Nat. Bur. Stand. (U.S.), Tech. Note 495, 45 pages (Sept. 1969), 50 cents.

Bullis, W. M., Thurber, W. R., Pyke, T. N., Jr., Ulmer, F. H., Koenig, A. L., Use of a time-shared computer system to control a Hall effect experiment, Nat. Bur. Stand. (U.S.), Tech. Note 510, 46 pages (Oct. 1969), 50 cents.

Burke, P. G., Cooper, J. W., Ormonde, S., Low energy scattering of electrons by helium, Phys. Rev. **183,** No. 1, 245–264 (July 5, 1969).

Carpenter, B. S., Cheek, C. H., Trace determination of uranium in biological material by fission track counting, Anal. Chem. **42,** No. 1, 121–123 (Jan. 1970).

Carroll, J. J., Melmed, A. J., Ellipsometry-LEED study of the adsorption of oxygen on (Oll) tungsten, Surface Sci. **16,** 251–264 (Aug. 1969).

Colwell, J. H., Molecular reorientation and nuclear spin conversion in the solid deuteromethanes at low temperature, J. Chem. Phys. **51,** No. 9, 3820–3832 (Nov. 1, 1969).

Colwell, J. H., The performance of a mechanical heat switch at low temperatures, Rev. Sci. Instr. **40,** No. 9, 1182–1186 (Sept. 1969).

Cooper, M. J., Vicentini-Missoni, M., Joseph, R. I., Linear parametrization of the equation of state near the critical point, Phys. Rev, Letters **23,** No. 2, 70–73 (July 14, 1969).

Coxon, B., Studies of ^{15}N-labeled amino sugars. Synthesis and spectroscopy of derivatives of 6-amino-6-deoxy-D-glucose-6-^{15}N, Carbohydrate Res. **11,** 153–155 (1969).

Crawford, M. L., A new RF–DC substitution calorimeter with automatically controlled reference power, IEEE Trans. Instr. Meas. **IM–17,** No. 4, 378–384 (Dec. 1968).

Daney, D. E., Ludtke, P. E., Sindt, C. F., Slush hydrogen pumping characteristics using a centrifugal-type pump (Proc. 1968 Cryogenic Engineering Conf., Case Western Reserve Univ., Cleveland, Ohio, Aug. 19–21, 1968), Chapter in Advances in Cryogenic Engineering 14, Paper J–2, 438–445 (Plenum Press, Inc., New York, N.Y., 1969).

Darwent, B. deB., Bond dissociation energies in simple molecules, Nat. Bur. Stand. (U.S.), Nat. Stand. Ref. Data Ser. 31, 52 pages (Jan. 1970), 55 cents, SD Catalog No. C13.48:31.

Davis, H. A., Jr., Hastings, J. R., Waxman, M., An improved tool for refinishing the conical ends of high pressure tubing, Rev. Sci. Instr. **40,** No. 9, 1238 (Sept. 1969).

Davis, J. C., Radiation errors in air ducts under nonisothermal conditions using thermocouples, thermistors, and a resistance thermometer, Nat. Bur. Stand. (U.S.), Bldg. Sci. Ser. 26, 14 pages (November 1969), 25 cents, SD Catalog No. C13.29/2:26.

Dean, J. W., Brennan, J. A., Mann, D. B., Cryogenic flow research facility of the National Bureau of Standards (Proc. 1968 Cryogenic Engineering Conf., Case Western Reserve Univ., Cleveland, Ohio, Aug. 19–21, 1968), Chapter in Advances in Cryogenic Engineering 14, Paper H–1, 299–305 (Plenum Press, Inc., New York, N.Y., 1969).

DeLancey, G. B., Analysis of multicomponent diaphragm cell data, J. Phys. Chem. **73,** No. 5, 1591–1593 (May 1969).

DeVoe, J. R., Editor, Radiochemical Analysis Section: Mössbauer spectroscopy, nuclear chemistry, nuclear instrumentation, radioisotrope techniques. Summary of activities, July 1968 to June 1969, Nat. Bur. Stand. (U.S.), Tech. Note 501, 149 pages (Feb. 1970), $1.25, SD Catalog No. C13.46:501.

Diamond, J. J., Editor, Bibliography on the high temperature chemistry and physics of materials, July, August, Sept. 1969, Nat. Bur. Stand. (U.S.), Spec. Publ. 315–3, 90 pages (Oct. 1969), $1.00.

Dickens, B., Brown, W. E., The crystal structures of $CaNa_2(CO_3)_2 \cdot 5H_2O$, synthetic gaylussite and $CaNa_2(CO_3)_2 \cdot 2H_2O$, synthetic pirssonite, Inorg. Chem. **8,** No. 10, 2093–2103 (Oct. 1969).

Donovan, R. J., Husain, D., Bass, A. M., Braun, W., Davis, D. D., Kinetic spectroscopic studies of $Cl(3p^5\,^2p_{3/2}, 1/2)$ in the vacuum ultraviolet, J. Chem. Phys. **50,** No. 9, 4115–4116 (May 1, 1969).

Drechsel, D., Maximon, L. C., Warner, R. E., Noncoplanar $p-p$ bremsstrahlung calculations at 48 and 30 MeV, Phys. Rev. **181,** No. 4, 1720–1722 (May 20, 1969).

Duerst, R. W., Kokoszka, G. F., Hyperfine fields in dimeric metal ion complexes, J. Chem. Phys. **51,** No. 4, 1673–1674 (Aug. 15, 1969).

Durst, R. A., Editor, Ion-selective electrodes. Proceedings of a symposium held at the National Bureau of Standards, Gaithersburg, Maryland, January 30–31, 1969, Nat. Bur. Stand. (U.S.), Spec. Publ. 314, 474 pages (November 1969), $3.50, SD Catalog No. C13.10:314.

Edmonds, D. K., Hork, J., Cavitation in liquid cryogens (Proc. 1968 Cryogenic Engineering Conf., Case Western Reserve Univ., Cleveland, Ohio, Aug. 19–21, 1968), Chapter in Advances in Cryogenic Engineering 14, Paper G–4, 274–282 (Plenum Press, Inc., New York, N.Y., 1969).

Eisele, J. W., Technical Standard Coordinator, Uncored slab urethane foam for bedding and furniture cushioning, Nat. Bur. Stand. (U.S.), Prod. Stand. 13–69, 6 pages (Dec. 1969), 10 cents, SD Catalog No. C13.20/2:13–69.

Engen, G. F., An introduction to the description and evaluation of microwave systems using terminal invariant parameters, Nat. Bur. Stand. (U.S.), Monogr. 112, 27 pages (October 1969), 35 cents.

Evanson, K. M., Microwave magnetic-dipole transitions between excited electronic states of CN, Phys. Rev. **178,** No. 1, 1–4 (Feb. 5, 1969).

Fatiadi, A. J., Acylation of tetrahydroxy-*p*-benzoquinone, J. Chem. Eng. Data **13,** No. 4, 591–593 (Oct. 1969).

Fatiadi, A. J., Novel iodination of the γ-methylene group in 1,3-diphenylpropane-1,3-dione and related β-diketones with periodic acid, Chem. Commun., p. 11 (1970).

Fenstermaker, C. A., McCrackin, F. L., Errors arising from surface roughness in ellipsometric measurement of the refractive index of a surface, Surface Sci. **16,** 85–96 (1969).

Flynn, T. M., Birmingham, B. W., Cryogenics and national goals (Proc. 1968 Cryogenic Engineering Conf., Case Western Reserve Univ., Cleveland, Ohio, Aug. 19–21, 1968), Chapter in Advances in Cryogenic Engineering 14, Paper A–1, 1–12 (Plenum Press, Inc., New York, N.Y., 1969).

Frederick, N. V., A low loss sliding short of limited travel for precision coaxial transmission lines, Proc. **IEEE 56,** No. 12, p. 2188 (Dec. 1968).

Furukawa, G. T., Automation problems in thermometry and calorimetry (Proc. 2nd & 3rd Japanese Calorimetry Conf., Osaka, Japan, Nov. 27–28, 1967), Chapter 1 in Calorimetry, Thermometry, and Calorimetry, Y. Otsubo, H. Kanbe, and S. Seki, eds., 2, 1–35 (Kagaku Gijitsu-Sha, Tokyo, Japan, 1969).

Gadzuk, J. W., Band-structure effects in the field-induced tunneling of electrons from metals, Phys. Rev. **182,** No. 2, 416–426 (June 10, 1969).

Gadzuk, J. W., Many-body tunneling-theory approach to field emission of elctrons from solids, Surface Sci. **15,** 466–482 (1969).

Garvin, D., Suggestion: the minireview, Phys. Today **22,** No. 10, 13–15 (Oct. 1969).

Gatterer, L. E., Clock synchronization experiments at VHF utilizing the ATS–1 and ATS–3 transponders, Natl. Aeronaut. Space Admin. Report ATS Tech. Summary (National Aeronautics and Space Administration, Greenbelt, Md., Feb. 16, 1968).

Geller, S. B., Mantek, P. A., Cleveland, N. G., Standard reference materials: Calibration of NBS secondary standard magnetic tape (computer amplitude reference) using the reference tape amplitude measurement "Process A", Nat. Bur. Stand. (U.S.), Spec. Publ. 260–18, 41 pages (Nov. 1969), 50 cents.

Gorden, T., Jr., Rebbert, R. E., Ausloos, P., Rare gas resonance lamps, Nat. Bur. Stand. (U.S.), Tech. Note 496, 55 pages (Oct. 1969), 60 cents.

Greenfeld, S. H., Natural weathering of mineral stabilized asphalt coatings on organic felt, Nat. Bur. Stand. (U.S.), Bldg. Sci. Ser. 24, 17 pages (Oct. 1969), 30 cents.

Gugeler, A. L., Ballard, D. B., The aluminum-porcelain enamel interface as observed by electron microscopy, Am. Ceramic Soc. Bull. **48,** No. 9, 842–845 (Sept. 1969).

Hall, L. A., A bibliography of thermophysical properties of air from 0 to 300 K, Nat. Bur. Stand. (U.S.), Tech. Note 383, 121 pages (Oct. 1969), $1.25, SD Catalog No. C13. 46:383.

Halldane, J. F., Visual responses of travelers in urban environments (Proc. Symp. Man-Machine Systems, Cambridge, England, Sept. 8–12, 1969), Chapter in Transport Systems Vehicle Control 2, Paper No. 8, 17 pages (1969).

Hardy, S. C., Coriell, S. R., Morphological stability of cylindrical ice crystals, J. Crystal Growth 5, 329–337 (1969).

Heaton, H. T., II, Use of an on-line computer in neutron time-of-flight measurements, Nat. Bur. Stand. (U.S.), Tech. Note 515, 31 pages (Jan. 1970), 35 cents, SD Catalog No. C13.46:515.

Hendricks, R. C., Simoneau, R., Smith, R. V., Heat transfer with a near-critical fluid, Nat. Aeronaut. Space Admin. Report TNX 52612 (National Aeronautics and Space Administration, Greenbelt, Maryland, June 16, 1969).

Herron, J. T., Huie, R. E., Rates of reaction of atomic oxygen. II. Some C_2 to C_8 alkanes, J. Phys. Chem. **73,** No. 10, 3327–3337 (Oct. 1969).

Hiza, M. J., Duncan, A. G., Equilibrium gas-phase compositions of ethane and ethylene in binary mixtures with helium and neon below 150° K and a correlation for deviations from the geometric

mean combining rule (Proc. 1968 Cryogenic Engineering Conf., Case Western Reserve Univ., Cleveland, Ohio, Aug. 19–21, 1968), Chapter in Advances in Cryogenic Engineering 14, Paper B–1, 30–40 (Plenum Press, Inc., New York, N.Y., 1969).

Hudson, G. E., Allan, D. W., Barnes, J. A., Hall, R. G., Lavanceau, J. D., Winkler, G. M. R., A coordinate frequency and time system, Proc. 23rd Frequency Control Symp., Atlantic City, N.J., May 6–8, 1969, p. 249 (1969).

Hudson, G. E., Barnes, J. A., Clock error statistics as a renewal process, Proc. 22nd Frequency Control Annual Symp., Atlantic City, N.J., Apr. 22–24, 1968, pp. 384–413 (1968).

Hummer, D. G., Non-coherent scattering—VI. Solutions of the transfer problem with a frequency-dependent source function, Mon. Not. R. astr. Soc. **145,** 95–120 (1969).

Isbell, H. S., Fatiadi, A. J., Phenylhydrazone-phenylazo tautomerism. Part III. Reactions of phenylformazans and certain bis(phenylhydrazaones) with strong acids, Carbohydrate Res. **11,** 303–311 (1969).

Isbell, H. S., Frush, H. L., Wade, C. W. R., Hunter, C. E., Transformations of sugars in alkaline solution, Carbohydrate Res. **9,** 163–175 (1969).

Jacox, M. E., Milligan, D. E., Matrix-isolation study of the infrared and ultraviolet spectra of several first-series transition-metal dichlorides, J. Chem. Phys. **51,** No. 9, 4143–4155 (Nov. 1, 1969).

Jefferson, C. F., Baker, C. K., Mechanism of electrical conductivity in nickel-iron ferrite, IEEE Trans. Magnetics **MAG–4,** No. 3, 460–464 (Sept. 1968).

Jellison, J. C., Collier, R. S., Fluid phase and temperature measurement with a single sensor (Proc. 1968 Cryogenic Engineering Conf., Case Western Reserve Univ., Cleveland, Ohio, Aug. 19–21, 1968), Chapter in Advances in Cryogenic Engineering 14, Paper H–5, 322–330 (Plenum Press, Inc., New York, N.Y., 1969).

Jennings, D. A., West, E. D., Evenson, K. M., Rasmussen, A. L., Simmons, W. R., Laser power and energy measurements, Nat. Bur. Stand. (U.S.), Tech. Note 382, 64 pages (Oct. 1969), 65 cents.

Johnson, V. J., A memoir of Russell B. Scott (Proc. CRYO/68 Conf., Chicago, Illinois, June 9–12, 1968), Chapter in Applications of Cryogenic Technology, R. W. Vance and H. Weinstock, eds., Introduction, pp. 1–11 (Tinnon-Brown, Inc., Los Angeles, California, 1969).

Joiner, B. L., The median significance level and other small sample measures of test efficacy, J. Am. Statist. Assoc. **64,** 971–985 (Sept. 1969).

Kamper, R. A., Possible contribution of superconducting devices to nuclear magnetic resonance detection, J. Appl. Phys. **40,** No. 5, 2163 (Apr. 1969).

Kamper, R. A., Mullen, L. O., Sullivan, D. B., Some applications of the Josephson effect, Nat. Bur. Stand. (U.S.), Tech. Note 381, 63 pages (Oct. 1969), 65 cents.

Kidnay, A. J., Hiza, M. J., Dickson, P. F., The kinetics of adsorption of methane and nitrogen from hydrogen gas (Proc. 1968 Cryogenic Engineering Conf., Case Western Reserve Univ., Cleveland, Ohio, Aug. 19–21, 1968), Chapter in Advances in Cryogenic Engineering 14, Paper B–2, 41–48 (Plenum Press, Inc., New York, N.Y., 1969).

Kieffer, L. J., The reliability of property data, or, whose guess shall we use?, J. Chem. Doc. **9,** No. 3, 167–168 (Aug. 1969).

Koeser, R. L., Report Editor, Report of the 54th national conference on weights and measures 1969, Nat. Bur. Stand. (U.S.), Spec. Publ. 318, 270 pages (Nov. 1969), $1.50, SD Catalog No. C13.6/3:318.

Knoeck, J., Vibrational spectrometric and electrochemical evidence for lanthanum(III)-nitrate complexes in aqueous solution, Anal. Chem. 41, No. 14, 2069–2071 (Dec. 1969).

Knoeck, J., Taylor, J. K., Aqueous boric acid-borate-mannitol equilibria, Anal. Chem. **41,** No. 13, 1730–1734 (Nov. 1969).

Kropschot, R. H., Thermal diffusivity of powder insulation (Proc. 1968 Cryogenic Engineering Conf., Case Western Reserve Univ., Cleveland, Ohio, Aug. 19–21, 1968), Chapter in Advances in Cryogenic Engineering 14, Paper F–4, 224–229 (Plenum Press, Inc., New York, N.Y., 1969).

Kuriyama, M., X-ray diffraction from a crystal containing isolated imperfections, Acta Cryst. **A,** No. 25, 682–693 (Nov. 1969).

Kushner, L. M., The NBS contribution to technological measurements and standards, Mater. Res. Std. **9,** No. 10, 8–10 (Oct. 1969).

LaFleur, P. D., Determination of molybdenum in steels and biological materials by activation analysis, Radiochem. Radioanal. Letters **1,** No. 3, 225–229 (July 1969).

LaVilla, R. E., Mendlowitz, H., Optical properties of germanium, J. Appl. Phys. **40,** No. 8, 3297–3300 (July 1969).

Logan, H. L., Studies of hot salt cracking of the titanium 8%-Al-1% Mo=1% V alloy (Proc. Ohio State Stress Corrosion Symp., Ohio State University, Columbus, Ohio, September 11–15, 1967), pp. 662–672 (National Association of Corrosion Engineers, Houston, Texas, 1969).

Lutz, G. J., LaFleur, P. D., Determination of yttrium in rare earths by photon activation analysis, Talanta **16,** 1457–1460 (1969).

Madey, T. E., Yates, J. T., Jr., Temperature effects in desorption by electron impact: molecular vibrations in the chemisorbed layer, J. Chem. Phys. **51,** No. 3, 1264–1265 (Aug. 1, 1969).

Mahler, R. J., Interaction of Gunn domains with the Ga^{71} nuclei in GaAs, Appl. Phys. Letters **14,** No. 9, 277–279 (May 1, 1969).

Maki, A. G., Jr., Vibrational spectroscopy, Ann. Rev. Phys. Chem. **20,** 273–292 (1969).

Mandel, J., A method for fitting empirical surfaces to physical or chemical data, Technometrics **11,** No. 3, 411–429 (Aug. 1969).

Margoshes, M., Remarks on linearization of characteristic curves in photographic photometry, Appl. Opt. **8,** No. 4, 818 (April 1969).

Margoshes, M., Rasberry, S. D., Application of digital computers in spectrochemical analysis—Computational methods in photographic microphotometry, Spectrochim. Act. **24B,** 497–513 (Jan. 14, 1969).

Marinenko, G., On the atomic weight of potassium, Talanta **16,** 1339–1340 (1969).

Marron, B. A., Tauber, S. J., Evolution of a general computer-based information system from pharmacological requirements, Proc. Conf. 32nd Annual Meeting of the American Society for Information Science, San Francisco, California, Oct. 1–4, 1969, 6, 223–230 (Greenwood Publ. Corp., Westport, Conn., 1969).

Martin, W. C., Sugar, J., Perturbations and coupling in the d^9sp configurations of Cu I, Zn II, Ag I, Cd II, and Tl III, J. Opt. Soc. Am. **59,** No. 10, 1266–1280 (Oct. 1969).

Mason, S. T., Dependence of the phase shift on energy and atomic number for electron scattering by atomic fields, Phys. Rev. **182,** No. 1, 97–101 (June 5, 1969).

Maximon, L. C., Comments on radiative corrections, Rev. Mod. Phys. **41,** No. 1, 193–204 (Jan. 1969).

McBee, C. L., Kruger, J., Ellipsometric-spectroscopy of films formed on metals in solution, Surface Sci. **16,** 340–352 (1969).

McCaa, W. D., Jr., Nahman, N. S., Frequency and time domain analysis of a superconductive coaxial line using the BCS theory, J. Appl. Phys. **40,** No. 5, 2098–2100 (April 1969).

Melmed, A. J., Surface characterization by ellipsometry low-energy electron diffraction, and field-electron microscopy (Proc. Conf. Third Battelle Memorial Institute Materials Science Colloquium, Kronberg, Germany, May 6–11, 1968), Chapter in Molecular Processes on Solid Surfaces, pp. 105–127 (McGraw-Hill Book Co., Inc., New York, N.Y., 1969).

Mendlowitz, H., Calculated transition array for the configurations $3d^2$-$3d4p$ in Ti III, Astrophys. J. **158,** 385–388 (Oct. 1969).

Menis, O., Analytical Coordination Chemistry Section: Summary of activities July 1968 to June 1969, Nat. Bur. Stand. (U.S.), Tech. Note 504, 120 pages (Oct. 1969), $1.00.

Mies, F. H., Resonant scattering theory of association reactions and unimolecular decomposition. I. A united theory of radiative and collisional recombination, J. Chem. Phys. **51,** No. 2, 787–797 (July 1969).

Miller, R. C., Prausnitz, J. M., Statistical thermodynamics of simple liquid mixtures. Henry's constants, Ind. Eng. Chem. Fundamentals **8,** No. 3, 449–452 (Aug. 1969).

Milligan, D. E., Jacox, M. E., Matrix-isolation study of the infrared and ultraviolet spectra of the free radical HCO. The hydrocarbon flame bands, J. Chem. Phys. **51,** No. 1, 277–288 (July 1, 1969).

Milligan, D. E., Jacox, M. E., Studies of the photoproduction of electrons in inert solid matrices. The electronic spectrum of the species C_2-, J. Chem. Phys. **51,** No. 5, 1952–1955 (Sept. 1, 1969). (Sept. 1, 1969).

Moore, C. E., Bibliography on the analyses of optical atomic spectra Section 4. $^{57}La-^{71}Lu$ and $^{89}Ac-^{99}Es$, Nat. Bur. Stand. (U.S.), Spec. Publ. 306, Section 4, 55 pages (Aug. 1969), 55 cents.

Motz, J. W., Olsen, H. A., Koch, H. W., Pair production by photons, Rev. Mod. Phys. **41,** No. 4, Part I, 581–639 (Oct. 1969).

Nahman, N. S., Superconducting transmission lines—communica-

tion and power, Proc. 1968 Summer Study on Superconducting Devices and Accelerators, Brookhaven National Laboratory, Upton, L.I., New York, June 10–July 19, 1968, BNL–50155, pp. 622–627 (1968).

Nemoto, T., Beatty, R. W., Fentress, G. H., A two-channel off-null technique for measuring small changes of attenuation, IEEE Trans. Microwave Theory Tech. **MTT**–17, No. 7, 396–397 (July 1969).

O'Connell, J. S., Prats, F., Isospin sum rules and the photodisintegration of the $A=3$ nuclei, Phys. Rev. **184,** No. 4, 1007–1012 (Aug. 20, 1969).

Paabo, M., Bates, R. G., Dissociation of deuteriocarbonate ion in deuterium oxide from 5 to 50°, J. Phys. Chem. **73,** No. 9, 3014–3017 (Sept. 1969).

Peiser, H. S., Wachtman, J. B., Jr., Jump rates for point defects in special positions held by a trapping centre of cubic symmetry, Chapter 12 in Physics of the Solid State, S. Balakrishna, M. Krishnamurty, and Rao B. Ramachandra, eds., pp. 207–222 (Academic Press, Inc., New York, N.Y., 1969).

Pfrang, E. O., Yokel, F. Y., Structural performance evaluation of a building system, Nat. Bur. Stand. (U.S.), Bldg. Sci. Ser. 25, 127 pages (Nov. 1969), $1.25.

Piermarini, G. J., Mighell, A. D., Weir, C. E., Block, S., Crystal structure of benzene II at 25 kilobars, Science **165,** 1250–1255 (Sept. 19, 1969).

Post, M. A., Determination of styrene-butadiene and styrene-acrylate resins in solvent type paint, J. Paint Technol. **41,** No. 537, 567–580 (Oct. 1969).

Qureshi, I. H., McClendon, L. T., LaFleur, P. D., Extraction studies of the group IIIB–VIIB elements and the lanthanides utilizing bis(2-ethylhexyl) orthophosphoric acid, Radiochim. Acta **12,** No. 2, 107–111 (Aug. 1969).

Reader, J., Optimizing Czerny-Turner spectrographs: A comparison between analytic theory and ray tracing, J. Opt. Soc. Am. **59,** No. 9, 1189–1196 (Sept. 1969).

Reed, R. P., Low-temperature mechanical properties of welded and brazed copper (Proc. 1968 Cryogenic Engineering Conf., Case Western Reserve Univ., Cleveland, Ohio, Aug. 19–21, 1968), Chapter in Advances in Cryogenic Engineering 14, Paper C–3, 83–87 (Plenum Press, Inc., New York, N.Y., 1969).

Reed, R. P., Schramm, R. E., Lattice parameters of martensite and austenite in Fe-Ni alloys, J. Appl. Phys. **40,** No. 9, 3453–3458 (Aug. 1969).

Richardson, A. C. B., Multiple scattering corrections for the associated-particle neutron time-of-flight technique, Nat. Bur. Stand. (U.S.), Tech. Note 499, 68 pages (Oct. 1969), 65 cents.

Roberts, D. E., Recrystallization and melting of partially melted stark rubber, Rubber Chem. Technol. **42,** 540–546 (Mar. 1969).

Robinson, R. A., An osopiestic vapor pressure study of the system potassium chloride-sodium chloride in deuterium oxide solution at 25°, J. Phys. Chem. **73,** No. 9, 3165–3166 (Sept. 1969).

Rockett, J. A., Planned operation of the National Bureau of Standards Fire Research and Safety Office, Fire J. **63,** No. 6, 41–43 (Nov. 1969).

Rush, J. J., Melveger, A. J., Lippincott, E. R., Laser-Raman spectra of PH_4I, PH_4Br, and PH_4Cl, J. Chem. Phys. **51,** No. 7, 2947–2955 (Oct. 1, 1969).

Ruthberg, S., Calibration to high precision in the medium vacuum range with stable environments and micromanometer, J. Vacuum Sci. Technol. **6,** No. 3, 401–412 (May/June 1969).

Rybicki, G. B., Hummer, D. G., Non-coherent scattering–V. Thermalization distances and their distribution function, Monthly Notices Roy. Astron. Soc. **144,** 313–323 (1969).

Schaffer, R., Editor, Organic Chemistry Section: Summary of activities July 1968 to June 1969, Nat. Bur. Stand. (U.S.), Tech. Note 507, 145 pages (Oct. 1969), $1.25.

Schwartz, R. B., Schrack, R. A., Heaton, H. T., A search for structure in the n-p scattering cross section, Physics Letters **30B,** No. 1, 36–38 (Sept. 1, 1969).

Scribner, B. F., Editor, Activities of the NBS Spectrochemical Analysis Section, July 1968 to June 1969, Nat. Bur. Stand. (U.S.), Tech. Note 502, 108 pages (Dec. 1969), $1.00, SD Catalog No. C13.46:502.

Shumaker, J. B., Jr., Popenoe, C. H., Arc measurement of some Ar II optical transition probabilities, J. Opt. Soc. Am. **59,** No. 8, 980–985 (Aug. 1969).

Simson, B. G., Radlinski, R. W., The accuracy of air tower pressure gages in suburban Washington, D.C., Nat. Bur. Stand. (U.S.), Tech. Note 512, 10 pages (Dec. 1969), 25 cents, SD Catalog No. C13.46:512.

Synder, N. S., Thermal conductance at the interface of a solid and helium II (Kaptiza conductance), Nat. Bur. Stand. (U.S.), Tech. Note 385, 90 pages (Dec. 1969), $1.00, SD Catalog No. C13.46:385.

Sparks, L. L., Powell, R. L., Hall, W. J., Progress on cryogenic thermocouples (Proc. 1968 Cryogenic Engineering Conf., Case Western Reserve Univ., Cleveland, Ohio, Aug. 19–21, 1968), Chapter in Advances in Cryogenic Engineering 14, Paper H–4, 316–321 (Plenum Press, Inc., New York, N.Y., 1969).

Stair, R., Measurement of natural ultraviolet radiation; historical and general introduction (Proc. Conf. Fourth International Biometeorological Congress, Rutgers University, New Brunswick, N.J., 1966), Chapter in The Biologic Effects of Ultraviolet Radiation, Frederick Urbach, ed., pp. 377–390 (Pergamon Press Inc., New York, N.Y., August 1969).

Standard Reference Materials, Catalog and price list of standard materials issued by the National Bureau of Standards, Nat. Bur. Stand. (U.S.), Spec. Publ. 260, 1969 Edition, 44 pages (July 1969) (Supersedes NBS Misc. Publ. 260–1968 Edition), 50 cents.

Stern, K. H., Weise, E. L., High temperature properties and decomposition of inorganic salts. Part 2. Carbonates, Nat. Bur. Stand. (U.S.), Nat. Stand. Ref. Data Ser. 30, 32 pages (Nov. 1969), 45 cents, SD Catalog No. C13.48:30.

Sukle, D. J., Wells, J. S., Fine structure in EPR of Mn^{2+} in zinc sulfate, Phys. Rev. **180,** No. 2, 445–450 (Apr. 10, 1969).

Taggart, H. E., Field strength and RFI standards at the National Bureau of Standards, Proc. IEEE Electromagnetic Compatibility Conf., Seattle, Washington, July 23–25, 1968, pp. 149–158 (1968).

Taylor, J. K., Editor, Microchemical Analysis Section: Summary of activities July 1968 to June 1969, Nat. Bur. Stand. (U.S.), Tech. Note 505, 140 pages (Oct. 1969), $1.25.

Thompson, B. A., LaFleur, P. D., Activation analysis for molybdenum in samples containing large amounts of tungsten, Anal. Chem. **41,** No. 13, 1888–1889 (Nov. 1969).

Tipson, R. S., Brady, R. F., Jr., Synthesis of the two D-2-pentuloses. New derivatives of D-*erythro*-pentulose, Carbohydrate Res. **10,** 549–563 (Aug. 1969).

Toots, J., Marton, L., Optical properties of antimony and bismuth in the far ultraviolet, J. Opt. Soc. Am. **59,** No. 10, 1305–1308 (Oct. 1969).

Venable, W. H., Jr., Shumaker, J. B., Jr., Observations of departures from equilibrium in a nitrogen arc, J. Quant. Spectr. Radiative Transfer 9, 1215–1226 (Sept. 1969).

Viezbicke, P. P., Why WWV moved to Colorado, Mark 5 **13,** No. 3, 14–16 (June 1968).

Waclawski, B. J., Hughey, L. R., Effect of heating on the photoelectron yield of polycrystalline tungsten in the vacuum ultraviolet, J. Opt. Soc. Am. **59,** 1494 (Nov. 1969).

Weir, C. E., Piermarini, G. J., Block, S., Instrumentation for single crystal x-ray diffraction at high pressures, Rev. Sci. Instr. **40,** No. 9, 1133–1136 (Sept. 1969).

West, E. D., Ishihara, S., Experimental evaluations of heater lead error in calorimetry, Rev. Sci. Instr. **40,** No. 10, 1356–1359 (Oct. 1969).

White, H. S., Jr., Editor, Congressional districts of the United States. Federal general data standard representations and codes, Nat. Bur. Stand. (U.S.), Fed. Info. Process. Stand. Publ. (FIPS Pub.) 9, 4 pages (Nov. 14, 1969), 10 cents, SD Catalog No. C13.52:9.

Wiederhorn, S. M., Fracture of sapphire, J. Am. Ceram. Soc. **52,** No. 9, 485–491 (Sept. 21, 1969).

Wiese, W. L., Bridges, J. M., Kornblith, R. L., Kelleher, D. E., Transition probabilities for prominent Ar I lines in the near infrared, J. Opt. Soc. Am. **59,** No. 9, 1206–1212 (Sept. 1969).

Wiese, W. L., Smith, M. W., Miles, B. M., Atomic transition probabilities. Volume II. Sodium through calcium, a critical data compilation, Nat. Stand. Ref. Data Ser., Nat. Bur. Stand. (U.S.), 22, 306 pages (Oct. 1969), $4.50.

Williams, R., The effects of extended high-temperature storage on the performance characteristics of several strain gage pressure transducers, Nat. Bur. Stand. (U.S.), Tech. Note 497, 22 pages (Oct. 1969), 30 cents.

Wilson, W. K., Hebert, R. L., Evaluation of the stability of record papers, Tappi **52,** No. 8, 1523–1529 (Aug. 1969).

Yates, J. T., Jr., Madey, T. E., Chemisorption on rhenium: N_2 and CO, J. Chem. Phys. **51,** No. 1, 334–337 (July 1, 1969).

Yates, J. T., Jr., Madey, T. E., Electron impact study of the γ-N_2

state chemisorbed on tungsten (Proc. Fourth International Materials Symp., Berkeley, California, June 1968), Chapter in The Structure and Chemistry of Solid Surfaces, Gabor A. Somorjai, ed., pp. 59–1, 59–27 (John Wiley and Sons, Inc., New York, N.Y., 1969).

**Publications for which a price is indicated are available by purchase from the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 20402 (foreign postage, one-fourth additional). The NBS non-periodical series are also available from the Clearinghouse for Federal Scientific and Technical Information, Springfield, Va. 22151. Reprints from outside journals and the NBS Journal of Research may often be obtained directly from the authors.*

U.S. GOVERNMENT PRINTING OFFICE : 1970 O-382-233

UNITED STATES
GOVERNMENT PRINTING OFFICE
DIVISION OF PUBLIC DOCUMENTS
WASHINGTON, D.C. 20402

OFFICIAL BUSINESS

POSTAGE AND FEES PAID
U.S. GOVERNMENT PRINTING OFFICE

JOURNAL OF RESEARCH of the National Bureau of Standards Vol. 74B, No. 2 Apr.–June 1970
B. Mathematical Sciences

Contents

	Page
Acoustic propagation and stability within an inviscid, heat-conducting fluid. J. E. McKinney and H. J. Oser.	**67**
On some indefinite integrals of confluent hypergeometric functions. Edward W. Ng and Murray Geller.	**85**
On complementary polar conical sets. Christoph Witzgall.	**99**
On contractive semigroups and uniform asymptotic stability. Philip R. Meyers.	**115**
Normal subgroups of the modular group. Leon Greenberg and Morris Newman.	**121**
Publications of the National Bureau of Standards.	**125**

Library of Congress Catalog Card Number: 62–4414

For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 20402. (Order by SD Catalog No. C 13.22/sec.B:74/2). Price $1.25 (single copy). Subscription price: $5.00 a year; $1.25 additional for foreign mailing.
UNITED STATES GOVERNMENT PRINTING OFFICE, WASHINGTON : 1970